13.003
Computational Geometry and Visualization

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Bézier curves and surfaces

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Good introductory books on Bézier/B-spline curves and surfaces are provided by Faux and Pratt [5], Mortenson [7], Ding and Davis [1], Rogers and Adams [10] and Nowacki et al. [8], while for more detailed descriptions on Bézier/B-spline representation the reader should refer to textbooks by Yamaguchi [11], Farin [2], Hoschek and Lasser [6] and Piegl and Tiller [9].

1 Bernstein polynomials

The *Bernstein polynomials* are defined as

$$B_{i,n}(t) = \frac{n!}{i!(n-i)!}(1-t)^{n-i}t^i, \qquad i = 0, \ldots, n, \tag{1}$$

and have several properties of interest.

- Positivity: $B_{i,n}(t) \geq 0, \qquad 0 \leq t \leq 1$ for all i and n.
- Partition of unity: $\sum_{i=0}^{n} B_{i,n}(t) = (1-t+t)^n = 1$ (by the *binomial* theorem)
- Symmetry:

$$B_{i,n}(t) = B_{n-i,n}(1-t) \tag{2}$$

- Recursion: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$ with $B_{i,n}(t) = 0$ for $i < 0$, $i > n$ and $B_{0,0}(t) = 1$
- Linear precision:

$$t = \sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) \tag{3}$$

which says the monomial t can be expressed as the weighted sum of Bernstein polynomials of degree n with coefficients evenly spaced in the interval [0,1].

- Degree elevation: The basis functions of degree n can be expressed in terms of those of degree $n+1$ [4] as:

$$B_{i,n}(t) = \left(1 - \frac{i}{n+1}\right) B_{i,n+1}(t) + \frac{i+1}{n+1} B_{i+1,n+1}(t), \quad i = 0, 1, \cdots, n \tag{4}$$

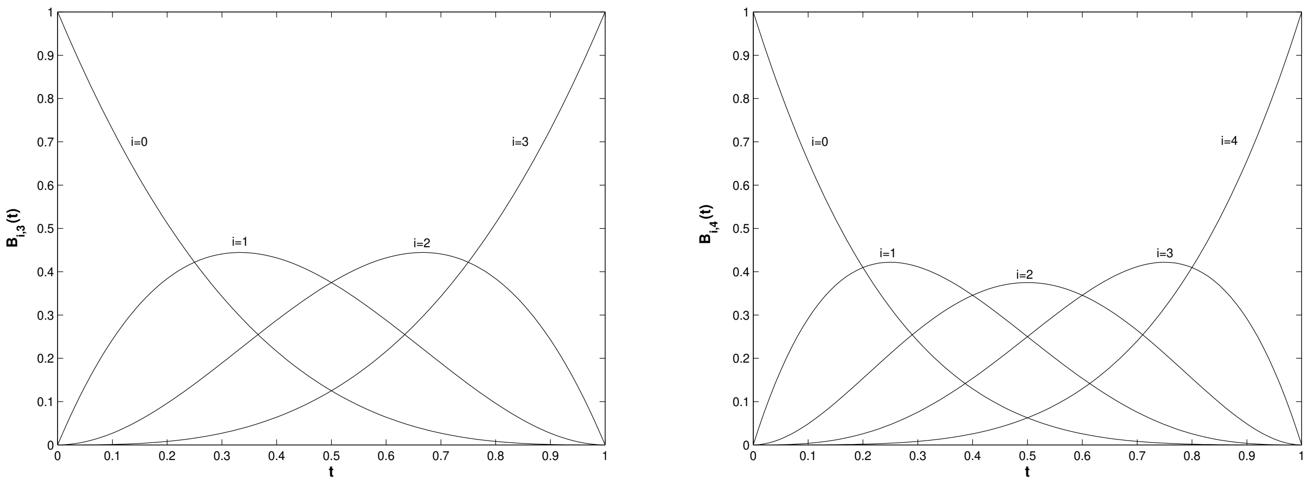
Or more generally in terms of basis functions of degree $n+r$ [4] as:

$$B_{i,n}(t) = \sum_{j=k}^{k+r} \frac{\begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} r \\ j-k \end{pmatrix}}{\begin{pmatrix} n+r \\ j \end{pmatrix}} B_{j,n+r}(t), \quad i = 0, 1, \cdots, n \tag{5}$$

Figure 1 shows the Bernstein polynomials of degree 3 and 4. The derivative of a Bernstein polynomial is

$$\frac{dB_{i,n}(t)}{dt} = n[B_{i-1,n-1}(t) - B_{i,n-1}(t)] \tag{6}$$

where $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$.



(a) (b)

Figure 1: Bernstein polynomials. (a) Degree three. (b) Degree four.

2 Arithmetic operations of polynomials in Bernstein form

Arithmetic operations between polynomials are often required for shape interrogation. Farouki and Rajan [4] provide formulae for such arithmetic operations of polynomials in Bernstein from. Let the two polynomials be $f(t)$ and $g(t)$ of degree m and n with Bernstein coefficients f_k^m and g_k^n as follows:

$$f(t) = \sum_{k=0}^{m} f_k^m B_{k,m}(t), \qquad g(t) = \sum_{k=0}^{n} g_k^n B_{k,n}(t), \qquad 0 \leq t \leq 1 \tag{7}$$

1. Addition and subtraction
   If the degrees of the two polynomials are the same, i.e. $m = n$, we simply add or subtract the coefficients

$$f(t) + g(t) = \sum_{k=0}^{m} (f_k^m \pm g_k^m) B_{k,m}(t). \tag{8}$$

   If $m > n$, we need to first degree elevate $g(t)$ m-n times using Equation (9) and then add or subtract the coefficients

$$f(t) + g(t) = \sum_{k=0}^{m} \left(f_k^m \pm \sum_{j=max(0,k-m+n)}^{min(n,k)} \frac{\binom{m-n}{k-j}\binom{n}{j}}{\binom{m}{k}} g_j^n \right) B_{k,m}(t). \tag{9}$$

2. Multiplication
   Multiplication of two polynomials of degree m and n yield a degree $m+n$ polynomial

$$f(t)g(t) = \sum_{k=0}^{m+n} \left(\sum_{j=max(0,k-n)}^{min(m,k)} \frac{\binom{m}{j}\binom{n}{k-j}}{\binom{m+n}{k}} f_j^m g_{k-j}^n \right) B_{k,m+n}(t). \tag{10}$$

3 Numerical condition of polynomials in Bernstein form

Polynomials in the Bernstein basis have better numerical stability under perturbation of their coefficients than in the power basis. We will introduce the concept of condition numbers for polynomial roots investigated by Farouki and Rajan [3].

Let us consider a polynomial $f(t)$ in the basis $\phi_k(t)$ with coefficients f_k:

$$f(t) = \sum_{k=0}^{n} f_k \phi_k(t) \tag{11}$$

Suppose we perturb a single coefficient f_j by δf_j, we have

$$\tilde{f}(t) = f_0\phi_0(t) + f_1\phi_1(t) + \ldots + (f_j + \delta f_j)\phi_j(t) + \ldots + f_n\phi_n(t) \tag{12}$$

or using (11)

$$\tilde{f}(t) = f(t) + \delta f_j \phi_j(t) \tag{13}$$

If $t + \delta t$ is a root of the perturbed polynomial $\tilde{f}(t)$, it satisfies

$$\tilde{f}(t+\delta t) = f(t+\delta t) + \delta f_j \phi_j(t+\delta t) = 0 \tag{14}$$

or

$$f(t+\delta t) = -\delta f_j \phi_j(t+\delta t) \tag{15}$$

Now let us Taylor expand about t_0, which is a root of $f(t)$, i.e. $f(t_0)=0$, on both sides of the above equation, we have:

$$\sum_{k=1}^{n} \frac{(\delta t)^k}{k!} \frac{d^k f}{dt^k}(t_0) = -\delta f_j \sum_{k=0}^{n} \frac{(\delta t)^k}{k!} \frac{d^k \phi_j}{dt^k}(t_0). \tag{16}$$

If t_0 is a simple root of $f(t)$, then $\dot{f}(t_0) \neq 0$, and in the limit of infinitesimal perturbations the above equation gives:

$$\lim_{\delta f_j \to 0} \frac{\delta t}{\frac{\delta f_j}{f_j}} = -\frac{f_j \phi_j(t_o)}{\dot{f}(t_0)} \tag{17}$$

The absolute value of the right hand side of the above equation

$$C = |f_j \phi_j(t_o)/\dot{f}(t_0)| \tag{18}$$

is called the *condition number* of the root t_0 with respect to the single coefficient f_j.

If t_0 is an m-fold root, $m \geq 2$, then a multiple-root condition number $C^{(m)}$ is defined in the form

$$C^{(m)} = \left(\frac{m!}{|\frac{d^m f(t_0)}{dt^m}|} \sum_{j=0}^{n} |f_j \phi_j(t_0)| \right)^{1/m}. \tag{19}$$

The following theorem is due to Farouki and Rajan [3].

Theorem 3.1 *For an arbitrary polynomial $f(t)$ with a simple root $t_0 \in [0,1]$, let $C_p(t_0)$ and $C_b(t_0)$ denote the condition numbers of the root in the power and Bernstein bases on $[0,1]$, respectively. Then $C_b(t_0) \leq C_p(t_0)$ for all $t_0 \in [0,1]$. In particular $C_b(0) = C_p(0) = 0$, while for $t_0 \in (0,1]$ we have the strict inequality $C_b(t_0) < C_p(t_0)$.*

As an illustration of the above theorem, let us consider a Wilkinson's polynomial where twenty real roots are equally distributed on $[0,1]$. Consider the polynomial with the linear distribution of real roots $t_0 = k/n, k = 1,2,...,n$, on the unit interval [0, 1] for $n = 20$:

$$f(t) = \prod_{k=1}^{20} (t - k/20).$$

The condition numbers for each root with respect to a perturbation in the single coefficient a_{19} are shown in Table 3 [3]. We can clearly observe that the condition numbers of the root in the Bernstein bases are several orders of magnitude smaller than in the power bases.

k	$C_p(x_0)$	$C_b(x_0)$
1	2.100×10^1	3.413×10^0
2	4.389×10^3	1.453×10^2
3	3.028×10^5	2.335×10^3
4	1.030×10^7	2.030×10^4
5	2.059×10^8	1.111×10^5
6	2.667×10^9	4.153×10^5
7	2.409×10^{10}	1.115×10^6
8	1.566×10^{11}	2.215×10^6
9	7.570×10^{11}	3.321×10^6
10	2.775×10^{12}	3.797×10^6
11	7.822×10^{12}	3.321×10^6
12	1.707×10^{13}	2.215×10^6
13	2.888×10^{13}	1.115×10^6
14	3.777×10^{13}	4.153×10^5
15	3.777×10^{13}	1.111×10^5
16	2.833×10^{13}	2.030×10^4
17	1.541×10^{13}	2.335×10^3
18	5.742×10^{12}	1.453×10^2
19	1.310×10^{12}	3.413×10^0
20	1.378×10^{11}	0

Table 1: Condition numbers for Wilkinson polynomial (Adapted from [3]).

4 Definition of Bézier curve and its properties

A *Bézier curve* is a spline curve that uses the Bernstein polynomials as a basis. A Bézier curve of degree n (order $n+1$) is represented by

$$\mathbf{r}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t), \qquad 0 \le t \le 1.$$

The coefficients, $\mathbf{b}_i$, are the *control points* or called *Bézier points* that determine the shape of the curve. Lines drawn between consecutive control points of the curve form the *control polygon*. A cubic Bézier curve together with its control polygon is shown in Figure 2. Bézier curves have the following properties:

- *Geometry invariance property*: Partition of unity property of the Bernstein polynomial assures the invariance of the shape of the Bézier curve under translation and rotation of its control points.

- *End points geometric property*:

  - The first and last control points are the endpoints of the curve. In other words, $\mathbf{b}_0 = \mathbf{r}(0)$ and $\mathbf{b}_n = \mathbf{r}(1)$.

  - The curve is tangent to the control polygon at the endpoints. This can be easily observed by taking the first derivative of a Bézier curve

$$\dot{\mathbf{r}}(t) = \frac{d\mathbf{r}(t)}{dt} = n \sum_{i=0}^{n-1} (\mathbf{b}_{i+1} - \mathbf{b}_i) B_{i,n-1}(t), \qquad 0 \le t \le 1. \tag{20}$$

    In particular we have $\dot{\mathbf{r}}(0) = n(\mathbf{b}_1 - \mathbf{b}_0)$ and $\dot{\mathbf{r}}(1) = n(\mathbf{b}_n - \mathbf{b}_{n-1})$. Equation (20) can be simplified by setting $\Delta\mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$:

$$\dot{\mathbf{r}}(t) = n \sum_{i=0}^{n-1} \Delta\mathbf{b}_i B_{i,n-1}(t), \qquad 0 \le t \le 1 \tag{21}$$

    The first derivative of a Bézier curve, which is called *hodograph*, is an another Bézier curve whose degree is lower than the original curve by one and has control points $\Delta\mathbf{b}_i$, $i = 0, \cdots, n-1$. Hodograph is useful in the study of intersection and other interrogation problems such as singularities and inflection points.

- *Convex hull property*: The convex hull is the intersection of all the convex sets containing all vertices or the intersection of the half spaces generated by taking three vertices at a time to construct a plane and having all other vertices on one side. The convex hull can also be conceptualized at the shape of a rubber band/sheet stretched taut over the polygon vertices. The entire curve is contained within the convex hull of the control points as shown in Figure 2. The convex hull property is useful in intersection problems (see Figure 3), in detection of absence of interference and in providing estimates of the position of the curve through simple bounds.

- *Variation diminishing property*:

– 2-D: The number of intersections of a straight line with a planar Bézier curve is no greater than the number of intersections of line with the control polygon. A line intersecting the convex hull of a planar Bézier curve may intersect the curve, be tangent to the curve, or not intersect the curve at all. It may not, however, intersect the curve more times than it intersects the control polygon. This property is illustrated in Figure 4.

– 3-D: The same relation holds true for a plane with a space Bézier curve.

From this property, we can roughly say that a Bézier curve oscillates less than its polygon, or in other words, the polygon's segments exaggerate the oscillation of the curve. This property is important in intersection algorithms and in detecting the *fairness* of Bézier curves.

- *Symmetry property*: If we renumber the control points as $\mathbf{b}^*_{n-i} = \mathbf{b}_i$, or in other words relabel from $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ to $\mathbf{b}_n, \mathbf{b}_{n-1}, \ldots, \mathbf{b}_0$ and using the symmetry property of the Bernstein polynomial (2) the following identity holds:

$$\sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t) = \sum_{i=0}^{n} \mathbf{b}^*_i B_{i,n}(1-t) \tag{22}$$

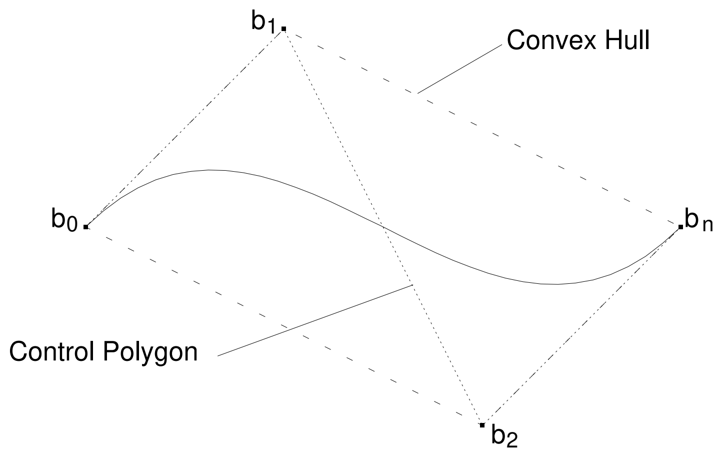


Figure 2: A cubic Bézier curve with control polygon.

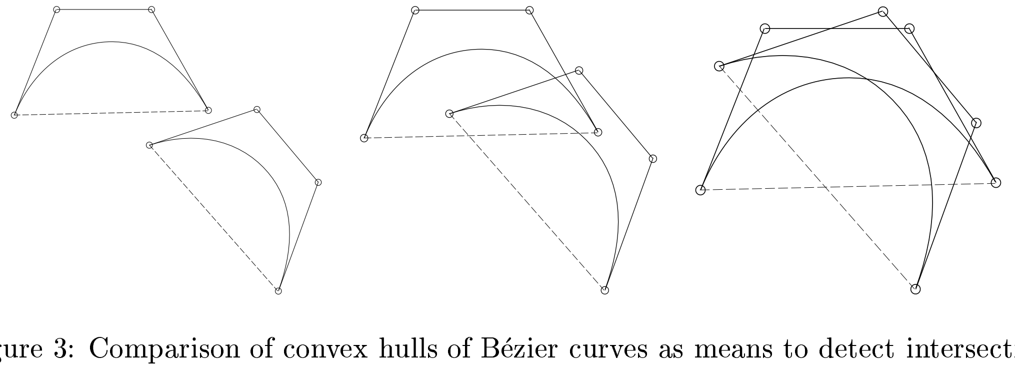


Figure 3: Comparison of convex hulls of Bézier curves as means to detect intersection.

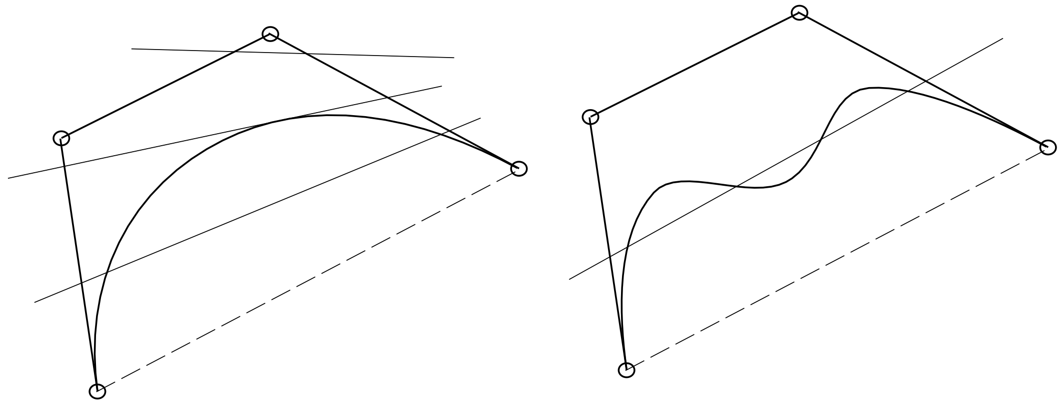


Figure 4: Variation diminishing property of a cubic Bézier curve.

5 Algorithms for Bézier curves

- *Evaluation and subdivision algorithm*: A Bézier curve can be evaluated at a specific parameter value t_0 and the curve can be split at that value using the *de Casteljau algorithm*, where [6]

$$\mathbf{b}_i^k(t_0) = (1-t_0)\mathbf{b}_{i-1}^{k-1} + t_0\mathbf{b}_i^{k-1}, \quad k = 1, 2, \ldots, n, \quad i = k, \ldots, n \tag{23}$$

  is applied recursively to obtain the new control points. The algorithm is illustrated in Figure 5, and has the following properties:

  - The values $\mathbf{b}_i^0$ are the original control points of the curve.
  - The value of the curve at parameter value t_0 is $\mathbf{b}_n^n$.
  - The curve can be represented as two curves, with control points $(\mathbf{b}_0^0,\ \mathbf{b}_1^1, \ldots,\ \mathbf{b}_n^n)$ and $(\mathbf{b}_n^n,\ \mathbf{b}_n^{n-1}, \ldots,\ \mathbf{b}_n^0)$.

- *Continuity algorithm*: Bézier curves can represent complex curves by increasing the degree and thus the number of control points. Alternatively, complex curves can be represented using composite curves, which can be formed by joining several Bézier curves end to end. If this method is adopted, the continuity between consecutive curves must be addressed.

  One set of continuity conditions are the *geometric* continuity conditions, designated by the letter G with an integer exponent. *Position continuity*, or G^0 continuity, requires the end-points of the two curves to coincide,

$$\mathbf{r}^a(1) = \mathbf{r}^b(0).$$

  The superscripts denote the first and second curves. *Tangent continuity*, or G^1 continuity, requires G^0 continuity and in addition the tangents of the curves to be in the same direction,

$$\dot{\mathbf{r}}^a(1) = \alpha_1\mathbf{t}$$
$$\dot{\mathbf{r}}^b(0) = \alpha_2\mathbf{t}$$

where $\mathbf{t}$ is the common unit tangent vector and α_1, α_2 are the magnitude of $\dot{\mathbf{r}}^a(1)$ and $\dot{\mathbf{r}}^b(0)$. G^1 continuity is important in minimizing stress concentrations in physical solids and preventing flow separation in fluids.

Curvature continuity, or G^2 continuity, requires G^1 continuity and in addition the center of curvature to move continuously past the connection point [5],

$$\ddot{\mathbf{r}}^b(0) = \left(\frac{\alpha_2}{\alpha_1}\right)^2 \ddot{\mathbf{r}}^a(1) + \mu\dot{\mathbf{r}}^a(1).$$

where μ is an arbitrary constant. G^2 continuity is important for aesthetic reasons and also for preventing fluid flow separation.

More stringent continuity conditions are the *parametric* continuity conditions, where C^k continuity requires the kth derivative (and all lower derivatives) of each curve to be equal at the joining point. In other words,

$$\frac{d^k\mathbf{r}^a(1)}{dt^k} = \frac{d^k\mathbf{r}^b(0)}{dt^k}.$$

The C^1 and C^2 continuity conditions for consecutive segments of a composite degree n Bézier curve can be stated as [11, 6]

$$\begin{aligned} h_{i+1}\left(\mathbf{b}_{ni} - \mathbf{b}_{ni-1}\right) &= h_i\left(\mathbf{b}_{ni+1} - \mathbf{b}_{ni}\right), \quad \text{and} && (24)\\ \mathbf{b}_{ni-1} + \frac{h_{i+1}}{h_i}(\mathbf{b}_{ni-1} - \mathbf{b}_{ni-2}) &= \mathbf{b}_{ni+1} + \frac{h_i}{h_{i+1}}(\mathbf{b}_{ni+1} - \mathbf{b}_{ni+2}) && (25) \end{aligned}$$

where, for the ith Bézier curve segment parameter t runs over the interval $[t_i, t_{i+1}]$, $h_i = t_{i+1} - t_i$ (see Figure 6 for the connection of cubic Bézier curve segments).

- *Degree elevation*: The degree elevation algorithm permits us to increase the degree and control points of a Bézier curve from n to $n+1$ without changing the shape of the curve. The new control points $\bar{\mathbf{b}}_i$ of the degree $n+1$ curve are given by

$$\bar{\mathbf{b}}_i = \frac{i}{n+1}\mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{b}_i, \quad i = 0, \ldots, n+1 \tag{26}$$

where $\mathbf{b}_{-1} = \mathbf{b}_{n+1} = \mathbf{0}$

6 Bézier surfaces

A tensor product surface is formed by moving a curve through space while allowing deformations in that curve. This can be thought of as allowing each control point $\mathbf{b}_i$ to sweep a curve in space. If this surface is represented using Bernstein polynomials, a Bézier surface (patch) is formed, with the following formula:

$$\mathbf{r}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{b}_{ij}B_{i,m}(u)B_{j,n}(v), \qquad 0 \le u, v \le 1.$$

Here, the set of straight lines drawn between consecutive control points $\mathbf{b}_{ij}$ is referred to as the *control net*. It is easy to see that boundary isoparametric curves ($u = 0$, $u = 1$, $v = 0$ and $v = 1$)

have the same control points as the corresponding boundary points on the net. An example of a bi-quadratic Bézier surface with its control net can be seen in Figure 7. Since a Bézier surface is a direct extension of univariate Bézier curve to its bivariate form, it inherits many of the properties of the Bézier curve described in Section 4 such as:

- Geometry invariance property
- End points geometric property
- Convex hull property

However, no variation diminishing property is known for Bézier surface patch.

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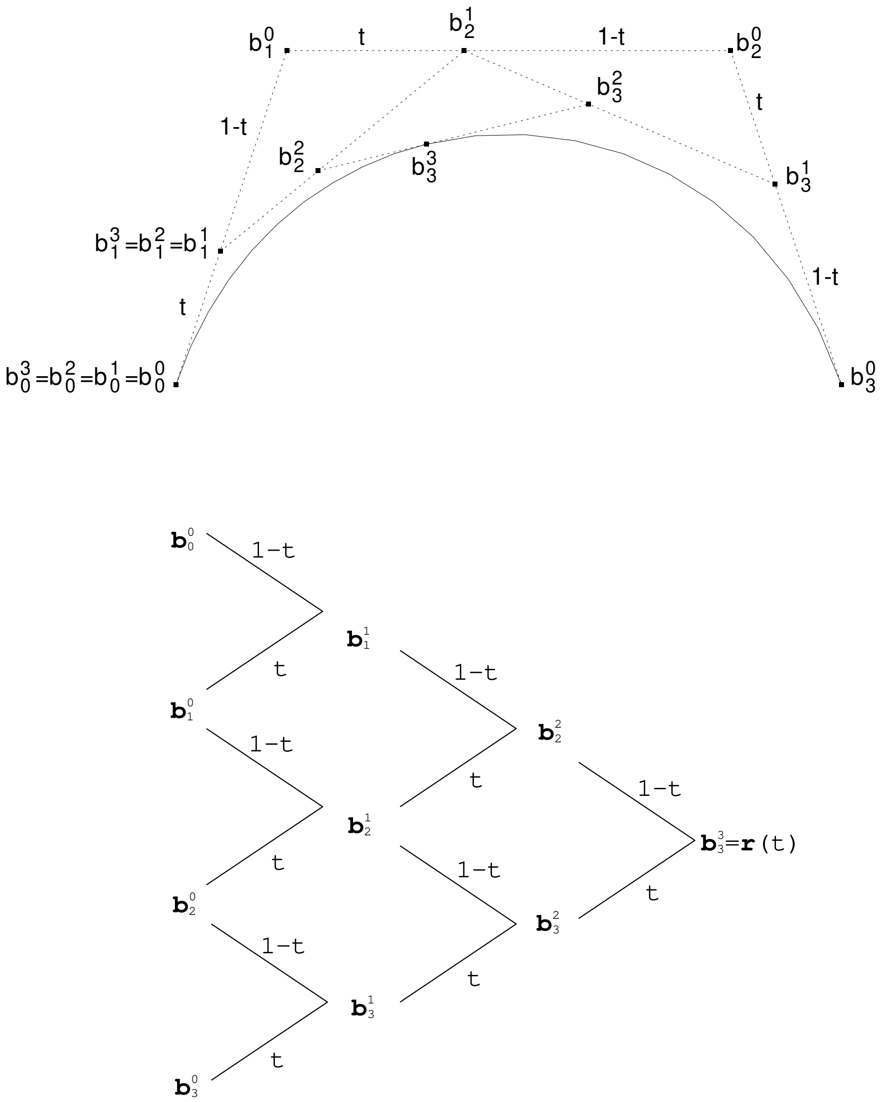

Figure 5: The de Casteljau algorithm.

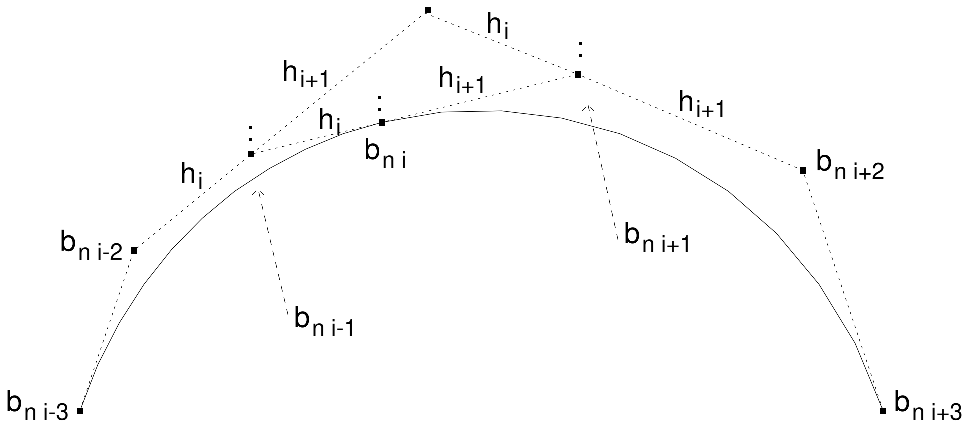

Figure 6: Continuity conditions.

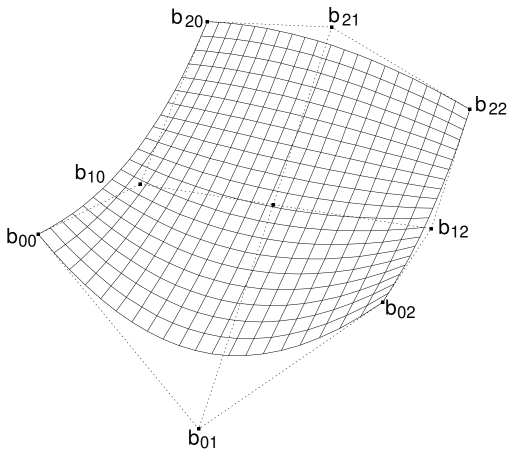

Figure 7: A Bézier Surface with Control Net.