

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Computational Geometry
Spring Term, 2003
Problem Set 1 on Differential Geometry

Issued: February 10, 2003
Due: February 24, 2003
Weight: 15% of total grade

Individual Effort

1. Show that the curvature of a planar curve is independent of the parametrization. Namely, if

$$\mathbf{r}(t) = [x(t), y(t)] \quad (1)$$

is the curve then a change of variables

$$t = w(u) \text{ with } w'(u) \neq 0 \quad (2)$$

does not affect the curvature. (Problem 18 in the textbook)

2. Let a curve \mathbf{X} be defined by

$$\mathbf{X}(t) = a \int \mathbf{g}(t) \times \mathbf{g}'(t) dt, \quad a = \text{const} \neq 0, \quad (3)$$

where $\mathbf{g}(t)$ is a vector function satisfying $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\mathbf{g}'\mathbf{g}''] \neq 0$. Show that the curvature and the torsion of the curve are $\kappa \neq 0$ and $\tau = 1/a$, respectively.

3. Find the parametric equation of a curve whose curvature κ and torsion τ are respectively

$$\kappa = \frac{a}{a^2 + b^2}, \quad \tau = \frac{b}{a^2 + b^2}, \quad (4)$$

where $a > 0$ and b are constants.

4. A curve \mathbf{C}_1 is called an *involute* of a given curve \mathbf{C} if tangents of \mathbf{C} are normal to \mathbf{C}_1 . The curve \mathbf{C} is called an *evolute* of \mathbf{C}_1 . Show that the curvature κ_1 of \mathbf{C}_1 is given by

$$\kappa_1^2 = \frac{\kappa^2 + \tau^2}{\kappa^2(c - s)^2}, \quad (5)$$

where c is a constant, s is the arc length of C measured from a fixed point on C , and κ and τ are the curvature and torsion of C .

5. Let E, F, G be the coefficients of the first fundamental form of a regular surface $\mathbf{R} = \mathbf{R}(u, v)$. Let $f(u, v) = c$ and $g(u, v) = d$ be two families of regular curves defined in the parameter space $u - v$ of the surface with images in 3D space obtained for various constants c and d . Prove that the 3D images of these two families of curves are orthogonal (i.e. whenever two curves of distinct families meet, their tangents are orthogonal) if and only if

$$Ef_v g_v - F(f_u g_v + f_v g_u) + Gf_u g_u = 0 \quad (6)$$

where $E = \mathbf{R}_u \cdot \mathbf{R}_u$, $F = \mathbf{R}_u \cdot \mathbf{R}_v$, $G = \mathbf{R}_v \cdot \mathbf{R}_v$, and subscripts u, v denote partial derivatives.

6. Consider a torus parametrized as follows:

$$\mathbf{r}(u, v) = [(R + a \cos u) \cos v, (R + a \cos u) \sin v, a \sin u] \quad (7)$$

where $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$, and R and a are constants such that $R > a$. Derive formulae for the Gauss, mean and principal curvatures. Sketch the torus and subdivide it into hyperbolic, parabolic and elliptic regions. In a follow-up sketch illustrate the lines of curvature of the torus. Explain the above subdivision and sketches. (Problem 17 in the textbook)

7. Show that the surface area on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is given by the integral

$$A = \int \int_W \sqrt{1 + f_u^2 + f_v^2} du dv, \quad (8)$$

where W is the parameter domain, and $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 the unit coordinate vectors.

8. Show that the second fundamental form on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is

$$II = (f_u^2 + f_v^2 + 1)^{-\frac{1}{2}} [f_{uu} du^2 + 2f_{uv} du dv + f_{vv} dv^2], \quad (9)$$

where $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 are the unit coordinate vectors.

9. Show that the principal curvatures of the surface $f(x, y, z) = x \sin(z) - y \cos(z) = 0$ are $\pm(x^2 + y^2 + 1)^{-1}$.

10. Consider the parametrized surface

$$\mathbf{r}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right). \quad (10)$$

Show that

- (a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, F = 0. \quad (11)$$

- (b) The coefficients of the second fundamental form are

$$L = 2, M = -2, N = 0. \quad (12)$$

- (c) The principal curvatures are

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}. \quad (13)$$