MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Computational Geometry
Spring Term, 2003
Problem Set 4 on Non-Linear Solver and Offsets

Issued: April 2, 2003
Due: April 23, 2003
Weight: 25% of total grade

Individual Effort

Problem 1: Planar parametric curve intersections (15% of total grade)

Write a program which determines all intersections of two integral planar cubic Bézier curves as accurately as possible, given the control points of the two curves as input and the desired accuracy in parametric and Cartesian space. Your program should report the parametric values of all intersection points as well as the corresponding Cartesian coordinates. Give three examples to show that your program works accurately. Your examples should include several transversal and near "tangent" intersections (carefully constructed). Transversal intersections are intersections at which the curves meet with their tangents at those points having an angle considerably different from zero. The implementation language can be C or C++ or any other language you are comfortable with.

Hint : A good approach involves binary subdivision using the de Casteljau algorithm and bounding box. In addition to bounding box comparisons, you may also consider using the concept of hodograph of a planar Bézier curve. If the angular sectors of the hodographs of the two Bézier curves (translated so they have the same apex) do not intersect, then the two curves intersect in at most one point. This criterion can be used to distinguish single transversal intersections from other more complex cases.

Problem 2: Offsets of Curves (5% of total grade)

Let $\mathbf{r}(s)$ be a planar, closed and convex curve (e.g. a circle, an ellipse, etc.) where the arc length s varies in the range $[0, l]$ so that the length of the curve is l. Let

$$\hat{\mathbf{r}}(s) = \mathbf{r}(s) + d\mathbf{n}(s)$$

be another curve, where d is a positive distance and $\mathbf{n}(s)$ is the unit normal vector of the curve $\mathbf{r}(s)$ defined by $\mathbf{n} = \mathbf{t} \times \mathbf{e}_z$. $\hat{\mathbf{r}}$ is called the interior *normal offset* at distance d.

a. Show that the total length of the curve $\hat{\mathbf{r}}(s)$ exceeds the total length of the curve $\mathbf{r}(s)$ by $2\pi d$.

b. Show that the area enclosed between the two curves is given by

$$A = d(l + \pi d)$$

c. Show that the curvatures of the two curves are related by

$$\hat{\kappa} = \frac{\kappa}{1 + d\kappa}$$

where κ is the curvature of $\mathbf{r}(s)$ and $\hat{\kappa}$ is the curvature of the offset curve $\hat{\mathbf{r}}(s)$.

d. Verify your results for questions a to c for a circle of radius R.

Problem 3: Regularity of Offsets of Planar Curves (5% of total grade)

This problem focuses on the identification of cusps, extraordinary points and self-intersections of offsets of planar curves. Consider the ellipse $x^2 + 4y^2 = 1$ or $x = cos\theta, y = \frac{1}{2}sin\theta$ and its offset at "distance" d, where d is any real number.

a. Determine all the values of θ for which there can be an extraordinary point on some offset of the ellipse and the values of d at such points. Sketch the offsets at all such values of d.

b. For what range of values of d, are offsets of the ellipse regular curves? Sketch a few such offset curves.

c. Determine a specific offset of the ellipse which includes several cusps and self-intersections but no extraordinary points. Infinite such cases exist. Give the parameter values and coordinates of these cusps and self-intersections.

(**Hint** : Note that self-intersections are on the axes of symmetry of the ellipse.)

Problem 4: Offsets of Surfaces [**OPTIONAL**, 5% EXTRA CREDIT]

Given a progenitor surface $\mathbf{r} = (u, v, -1.75u^2 + 2v^2)$, which is a hyperbolic paraboloid.

a. Find the unit normal vector $\hat{\mathbf{n}}$ of the offset surface for offset distance d in terms of $\mathbf{n}$, the unit normal vector of $\mathbf{r}$.

b. Explain when $\hat{\mathbf{n}}$ switches sign at $(0, 0, 0)$.

c. Assuming $d > 0$, what is the critical offset distance knowing that both principal curvatures have extrema at $(0, 0, 0)$?

d. When the offset distance is larger than the critical offset distance, a self-intersection occurs. Obtain the equations for the self-intersection curve.

(**Hint** : See Chapter 11 of book.)