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DEPARTMENTS OF OCEAN, MECHANICAL,
CIVIL & ENVIRONMENTAL AND AERONAUTICAL &
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Spring Term, 2000
Problem Set 1 on Differential Geometry

Issued: February 3, 2000
Due: February 24, 2000
Weight: 15% of total grade (individual effort)

1. Show that the curvature of a planar curve is independent of the parametrization. Namely, if
$$\mathbf{r}(t) = [x(t), y(t)] \tag{1}$$
is the curve then a change of variables
$$t = w(u) \quad with \quad w'(u) \neq 0 \tag{2}$$
does not affect the curvature.

2. Let a curve C be defined by
$$\mathbf{X}(t) = a \int \mathbf{g}(t) \times \mathbf{g}^{'}(t) dt, \quad a = const \neq 0,$$
where $\mathbf{g}(t)$ is a vector function satisfying $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\mathbf{g}^{'}\mathbf{g}^{''}] \neq 0$. Show that the curvature of the curve $\kappa \neq 0$ and $\tau = 1/a$.

3. Let E, F, G be the coefficients of the first fundamental form of a regular surface $\mathbf{R} = \mathbf{R}(u, v)$. Let $f(u, v) = c$ and $g(u, v) = d$ be two families of regular curves defined in the parameter space $u - v$ of the surface with images in 3D space obtained for various constants c and d. Prove that the 3D images of these two families of curves are orthogonal (i.e. whenever two curves of distinct families meet, their tangents are orthogonal) if and only if
$$Ef_v g_v - F(f_u g_v + f_v g_u) + G f_u g_u = 0 \tag{3}$$
where $E = \mathbf{R}_u \cdot \mathbf{R}_u, F = \mathbf{R}_u \cdot \mathbf{R}_v, G = \mathbf{R}_v \cdot \mathbf{R}_v$, and subscripts u, v denote partial derivatives.

4. Show that the surface area on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is given by the integral
$$A = \int\int_W \sqrt{1 + f_u^2 + f_v^2} du dv,$$
where W is the parameter domain and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the unit coordinate vectors.

5. Show that the second fundamental form on a Monge patch $\mathbf{X}(u,v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is

$$II = (f_u^2 + f_v^2 + 1)^{-1/2}[f_{uu}du^2 + 2f_{uv}dudv + f_{vv}dv^2],$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit coordinate vectors.

6. Consider a torus parametrized as follows:

$$\mathbf{r}(u,v) = [(R + a\ cos\ u)cos\ v, (R + a\ cos\ u)sin\ v, a\ sin\ u] \qquad (4)$$

where $0 \le u \le 2\pi, 0 \le v \le 2\pi$ and R , a are constants such that $R > a$. Derive formulas for the Gauss, mean and principal curvatures. Sketch the torus and subdivide it into hyperbolic, parabolic and elliptic regions. In a follow-up sketch illustrate the lines of curvature of the torus. Explain the above subdivision and sketches.

7. Show that the principal curvatures of the surface

$$f(x,y,z) = x \sin z - y \cos z = 0$$

are

$$\pm(x^2 + y^2 + 1)^{-1}.$$