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Problem Set 2 on B-Splines and NURBS
INDIVIDUAL EFFORT

Issued: Tuesday, 15 February 2000
Due: Tuesday, 7 March 2000
Weight: 25% of total grade

The following problems do not need computer programming.

1. A cubic planar Bézier curve:

$$\mathbf{R}(t) = \sum_{i=0}^{3} \mathbf{R}_i \mathbf{B}_{i,3}(t) \quad 0 \leq t \leq 1 \tag{1}$$

has the following control points

$$\mathbf{R}_0 = [0,0], \mathbf{R}_1 = [2,2], \mathbf{R}_2 = [3,2], \mathbf{R}_3 = [3,0].$$

A designer decides to subdivide (split) the curve at $t_0 = 1/2$ and $t_1 = 2/3$ in order to be able to modify the curve in the interval $[t_0, t_1]$ and generate a particular shape feature required by his design.

(a) Compute the coordinates of the control points of the three curve segments generated by the above subdivision.

(b) After the above subdivision, the middle segment of the three curve segments created by subdivision is permitted to be modified by changing the coordinates of its control points. Determine the conditions that the control points must obey so that this curve segment maintains position continuity at its ends with the end and the beginning of the other two curve segments.

(c) Assuming position continuity is maintained as in (b), determine the additional necessary conditions to maintain unit tangent vector continuity at the ends of the middle curve segment.

(d) Assuming conditions (b) and (c) are satisfied, determine the coordinates of the vertices of the polygonal boundary of the convex hull of the middle curve segment. You need to distinguish several special cases.

2. Derive the Ferguson (or monomial or power basis), the Hermite-Coons, and the Lagrange forms of curve $\mathbf{R}(t)$ of problem 1 prior to any subdivision. For the Lagrange form use interpolation at the nodes $\tau_i = i/3$, for $i = 0, 1, 2, 3$.

3. Derive the (uniform) B-spline form of curve $\mathbf{R}(t)$ of problem 1 prior to any subdivision. Make a rough plot of the curve (by hand) together with its Bézier and B-spline polygon illustrating the principal features of the curve (such as tangencies at the ends etc.). Compare the convex hulls of the curve in the Bézier and the B-spline form in terms of the area they enclose (i.e. determine the ratio of the two areas). Note: Transformation of a cubic curve from the Bézier to the B-spline form allows the user to *easily* attach additional cubic curve spans with second order derivative continuity without explicit use of the constraints of composite Bézier curves.

4. Show that the derivative of a Bézier curve (also called *hodograph*) of the form:

$$\mathbf{R}(t) = \sum_{i=0}^{n} \mathbf{R}_i B_{i,n}(t) \quad 0 \leq t \leq 1 \tag{2}$$

is given by:

$$\mathbf{R}'(t) = \sum_{i=0}^{n-1} n(\mathbf{R}_{i+1} - \mathbf{R}_i) B_{i,n-1}(t) \quad 0 \leq t \leq 1 \tag{3}$$

Sketch a cubic Bézier curve and its hodograph and their control polygons.

5. A cusp on a parametric curve $\mathbf{R}(t)$ occurs at $t = t_0$ when $\mathbf{R}'(t_0) = \mathbf{0}$. Can a non-planar cubic Bézier curve have a cusp? Give an example of a cubic Bézier curve with a cusp and identify the cusp.

6. Show how an explicit polynomial curve $y = y(x)$, where $a \leq x \leq b$ can be converted into a Bézier curve. Provide the control points of the resulting Bézier curve. Apply this to the case $y(x) = 1 + x + x^2 + x^3, a = 0, b = 2$, (also using reparametrization $t = x/2$ to make the interval $[0, 1]$).

7. Given a B-spline curve with a non-uniform knot vector, the control polygon of which is symmetric with respect to the y-axis, find if the curve is also symmetric about the y-axis.

8. Express the paraboloid $\vec{\mathbf{r}}(\phi, t) = (t\cos\phi, t\sin\phi, t^2)$, where $0 \leq \phi \leq \frac{\pi}{2}, 0 \leq t \leq 1$ in terms of a rational Bézier surface.

The following problems need computer programming.

9. Make plots of the B-spline basis functions of the following order n (degree $= n - 1$) and knot vector T:

   - n = 4, T = [0, 0, 0, 0, 1, 1, 1, 1]
   - n = 4, T = [0, 0, 0, 0, 2, 4, 4, 4, 4]
   - n = 4, T = [0, 0, 0, 0, 3, 3, 6, 6, 6, 6]
   - n = 4, T = [0, 0, 0, 0, 2, 2, 2, 6, 6, 6, 6]
   - n = 4, T = [0, 0, 0, 0, 1, 2, 3, 4, 6, 7, 7, 7, 7]
   - n = 3, T = [0, 0, 0, 3, 6, 9, 9, 9]
   - n = 2, T = [0, 0, 2, 4, 4]

   You may use *MATLAB* or *Tecplot* on Athena (or OpenGL or another package) for your plots.

10. Given a list of Cartesian points in 3-D space which represent a non-periodic curve, construct a cubic Bézier curve using least squares approximation of the points using Numerical Recipes routines. Also, construct a cubic B-spline curve with non-uniform knots using least squares approximation of these points using Numerical Recipes routines. A user should be able to access your program with an arbitrary number of points and coordinates of points. Include a simple visualization of the results as in problem 9.