Surface to Surface Intersections

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ABSTRACT

This paper presents an overview of surface intersection problems and focuses on the rational polynomial parametric/rational polynomial parametric surface intersection case including transversal and tangential intersections. Emphasis is placed on marching methods with a discussion of the problems with conventional tracing algorithms. An approach using a validated interval ordinary differential equation system solver is outlined and illustrated with examples, which offers significant advantages in robustness over conventional marching schemes.

Keywords: rounded interval arithmetic, boundary representation, parametric surfaces, singularity, tangency.

1. INTRODUCTION

Intersections are fundamental in CAD, CAM, computational geometry, geometric modeling and design, analysis and manufacturing applications [11,29]. Examples of intersection problems include: (1) Contouring of surfaces through intersection with a series of parallel planes or coaxial cylinders for visualization. (2) Numerical control machining (milling) involving intersection of generalized offset surfaces with a series of parallel planes, to create machining paths. (3) Representation of complex geometries in the Boundary Representation (B-rep) scheme using a process called boundary evaluation, in which the Boundary Representation is created by evaluating a Constructive Solid Geometry (CSG) model of the object. In this process, intersections of the surfaces of primitives must be found during Boolean operations (union, intersection, difference) between primitives.

When studying intersection problems, the type of curves and surfaces that we consider can be classified primarily into two types: (1) Rational polynomial parametric (RPP) and (2) Implicit algebraic (IA). Non-Uniform Rational B-Spline (NURBS) curves and surfaces can be subdivided into RPP curves and surfaces, and analyzed in a similar manner. A detailed treatment of intersection problems including general procedural curves and surfaces can be found in [31,30].

Among all types of intersections, the surface to surface (S/S) intersection is the most complicated problem. It can have various intersection components such as curve segments, points, loops and singular points. Therefore, any surface intersection algorithm should satisfy the robustness requirements imposed by solid and geometric modeling in (a) finding all the components of the intersection and (b) computing each component with high accuracy.

In this paper we only deal with S/S intersection problems of RPP/RPP type with emphasis on tracing the intersection curve robustly. General intersection problems are analyzed in [31,30].

This paper is structured as follows: Section 2 presents a classification of surface to surface intersection problems. In Section 3, a marching solution method is explained with formulation of transversal and tangential surface intersections. In Section 4, a robust marching method based on a validated ODE solver is introduced with examples illustrating the method. Section 5 concludes the paper.

2. CLASSIFICATION OF SURFACE TO SURFACE INTERSECTION PROBLEMS

An implicit algebraic surface is represented by a polynomial function defined as \( f(\mathbf{r}) = 0 \), where \( \mathbf{r} \) is the position vector of a point on the surface. The rational polynomial parametric type includes Bézier, rational Bézier, B-spline and NURBS surface patches, which are represented with two parameters \( u \) and \( v \) as \( \mathbf{r} = \mathbf{r}(u, v) \).
0 ≤ u, v ≤ 1. These surfaces are popular in CAD/CAM and geometric design, and NURBS surfaces are chosen as the standard format in industry. Depending on the surfaces involved in intersection, we have three distinct classes: IA/IA, RPP/IA and RPP/RPP. They are the most frequent surface to surface intersection problems.

2.1. IA/IA Surface Intersection

Implicit algebraic surface to implicit algebraic surface intersection is defined as follows:

\[ f(r) = 0 \cap g(r) = 0, \]

where \( f, g \) are polynomial functions. Here we have two equations in three unknowns \( r \).

A method for low order \( f, g \) is to eliminate one variable (e.g. \( z \)) to find projection of intersection curves on the plane of other two variables (e.g. \( x, y \)), then trace the algebraic curve and use the inversion algorithm to find \( z \). Intersections of low degree implicit algebraic surfaces are of special interest in the boundary evaluation of the Constructive Solid Geometry models. A more complete analysis of the special intersections of two quadric surfaces can be found in [25,37,42].

2.2. RPP/IA Surface Intersection

Rational polynomial parametric surface to implicit algebraic surface intersection is defined as follows:

\[ r(u,v) \cap f(r) = 0, \quad 0 ≤ u, v ≤ 1, \]

(2)

where \( r(u,v) = \left( \frac{X(u,v)}{W(u,v)}, \frac{Y(u,v)}{W(u,v)}, \frac{Z(u,v)}{W(u,v)} \right)^T \). This leads to four algebraic equations in five unknowns \( r = (x, y, z), u, v \). For the usual low degree surfaces \( f(r) \) and low degree patches \( r(u,v) \), we can substitute \( r(u,v) \) into \( f(r) = 0 \) to obtain an implicit algebraic curve in \( u, v \), see [16,17,32,33] for detailed treatment.

2.3. RPP/RPP Surface Intersection

Rational polynomial parametric surface to rational polynomial parametric surface intersection is defined as follows:

\[ r_1(\sigma,t) \cap r_2(u,v), \]

(3)

(0 ≤ \( \sigma, t ≤ 1 \), 0 ≤ u, v ≤ 1)

where \( r_1(\sigma,t) = \left( \frac{X_1(\sigma,t)}{W_1(\sigma,t)}, \frac{Y_1(\sigma,t)}{W_1(\sigma,t)}, \frac{Z_1(\sigma,t)}{W_1(\sigma,t)} \right)^T \) and \( r_2(u,v) = \left( \frac{X_2(u,v)}{W_2(u,v)}, \frac{Y_2(u,v)}{W_2(u,v)}, \frac{Z_2(u,v)}{W_2(u,v)} \right)^T \). Formulation involves setting \( r_1(\sigma,t) = r_2(u,v) \) which leads to three nonlinear polynomial equations in four unknowns \( \sigma, t, u, v \). This is an underconstrained system. This system can in principle be solved by the Interval Projected Polyhedron (IPP) algorithm [38]. However, as the solutions are typically not isolated points but curves, such approach is inefficient when small tolerances are used. Another method involves implicitization of \( r(\sigma,t) \) to the form \( f(r) = 0 \) and substitution of \( r = r_1(u,v) \) into \( f \) to reduce the problem to RPP/IA case for a low degree surface [17]. Heo et al. [10] developed an intersection algorithm for two ruled surfaces which performs more efficiently than those for general parametric surfaces.

There are three major techniques for solving RPP/RPP surface intersections: lattice methods, subdivision methods and marching methods. Detailed reviews can be found in [29,31,30]. In this paper, we focus on marching methods which are efficient in most cases and hence attractive if combined with other methods such as adaptive subdivision methods to locate starting points.

3. MARCHING METHODS

Marching methods involve generation of sequences of points of an intersection curve branch by stepping from a given point on the required curve in a direction prescribed by the local differential geometry [1,2,15,43]. Marching method formulates the surface intersection as an initial value problem (IVP) in the domain \( 0 ≤ \sigma, t, u, v ≤ 1 \). However, such methods are by themselves incomplete in that they require starting points (initial conditions) for every branch of the solution.

3.1. Computation of Starting Points

In order to identify all connected components of an intersection curve, a set of characteristic points on the intersection curve can be defined. Such a set may include border, turning and singular points of the intersection and provides at least one point on any connected intersection segment and identifies all singularities. For RPP/RPP surface intersections a more convenient set of such points sufficient to discover all connected components of the intersection, includes border and collinear normal points between the two surfaces. Collinear normal points provide points inside all intersection loops and all singular points [12]. Border points are points of the intersection at which at least one of the parametric variables \( \sigma, t, u, v \) takes a value equal to the border of the \( \sigma - t \) or \( u - v \) parametric domain. To compute border points, a piecewise rational polynomial curve to piecewise rational polynomial surface intersection capability is required, e.g.,
The normal vectors satisfy the following equations \(4\):  

\[
\sigma' = \frac{\text{det}(c', p, P(\sigma, t))}{\text{det}(p, P(\sigma, t))}, \quad \tau' = \frac{\text{det}(c', p, P(\sigma, t))}{\text{det}(p, P(\sigma, t))},
\]

where \(\text{det}\) denotes the determinant (see also [8]) and \(P(\sigma, t) = p_s \times p_r, \quad Q(u, v) = q_u \times q_v\), are the normal vectors of \(p\) and \(q\), respectively.

### 3.2.2. Tangential Intersection

When the two surfaces intersect tangentially, we cannot use Equation (8) since the denominator vanishes. In such cases we must find the marching direction in an alternate way [44].

The unit tangent vector \(c'(s)\) must lie on the common tangent plane of \(p(\sigma, t)\) and \(q(u, v)\). It can be defined using the linear combination of the partial derivatives \(p_s\), \(p_r\), \(q_u\) and \(q_v\) of each of the surfaces as follows:

\[
c'(s) = p_s \sigma' + p_r \tau' = q_u u' + q_v v'.
\]

Since the normal vectors of the surfaces at a point on the intersection are the same, both surfaces have the same normal curvature at that point in the direction \(c'(s)\) of the intersection curve. This implies that the second fundamental forms of both surfaces are equal, which can be expressed as follows:

\[
L''(\sigma')^2 + 2M''\sigma' \tau' + N''(\tau')^2 = L''(u')^2 + 2M''u' \tau' + N''(v')^2,
\]
where \( L^p, M^p, N^p \) and \( L^q, M^q, N^q \) are the second fundamental form coefficients of both surfaces.

This is a quadratic equation in \( \sigma', t', u', v' \). By taking the cross product of both sides of equation (9) with \( q \) and \( q \), and projecting the resulting equations onto the common surface normal vector \( N \) at a point on the intersection, \( u' \) and \( v' \) can be represented as the following linear combination of \( \sigma' \) and \( t' \):

\[
\begin{align*}
\mathbf{u}' &= a_{11} \mathbf{\sigma}' + a_{12} \mathbf{t}', \\
\mathbf{v}' &= a_{21} \mathbf{\sigma}' + a_{22} \mathbf{t}'.
\end{align*}
\]

(11)

(12)

where the coefficients \( a_{11}, a_{12}, a_{21}, \) and \( a_{22} \) are

\[
\begin{align*}
a_{11} &= \left( \mathbf{p}_x, \mathbf{q}_x \right) N \\
&= \frac{\det(\mathbf{p}_x, \mathbf{q}_x, N)}{\sqrt{E^q G^q - (F^q)^2}}, \\
a_{12} &= \left( \mathbf{q}_x, \mathbf{p}_x \right) N \\
&= \frac{\det(\mathbf{q}_x, \mathbf{p}_x, N)}{\sqrt{E^q G^q - (F^q)^2}}, \\
a_{21} &= \left( \mathbf{q}_x, \mathbf{p}_x \right) N \\
&= \frac{\det(\mathbf{q}_x, \mathbf{p}_x, N)}{\sqrt{E^q G^q - (F^q)^2}}, \\
a_{22} &= \left( \mathbf{p}_x, \mathbf{q}_x \right) N \\
&= \frac{\det(\mathbf{p}_x, \mathbf{q}_x, N)}{\sqrt{E^q G^q - (F^q)^2}}.
\end{align*}
\]

Here, \( E^q, G^q, F^q \) are the first fundamental form coefficients of the surface \( q \).

Substituting (11) and (12) into (10), then we obtain a quadratic equation of the form,

\[
\begin{align*}
b_{11} & = a_{11}^2 L^p + 2a_{11}a_{21}M^q + a_{21}^2 N^q - L^p, \\
b_{12} & = a_{11}a_{22}L^p + (a_{11}a_{22} + a_{21}a_{21})M^q + a_{21}a_{22}N^q - M^p, \\
b_{22} & = a_{22}^2 L^p + 2a_{22}a_{21}M^q + a_{22}^2 N^q - N^p.
\end{align*}
\]

There are four distinct cases to the solution of (13) depending upon the discriminant \( d = b_{11}^2 - b_{11}b_{22} \).

- \( d < 0 \): The surfaces have an isolated tangential contact point.
- \( d > 0 \): We have the phenomenon of branching, i.e. \( \mathbf{c}'(s) \) is not uniquely defined.
- \( d = 0 \) and \( b_{11}, b_{12}, b_{22} = 0 \): The intersection of surfaces \( p \) and \( q \) cannot be evaluated by this method or they have a contact of at least second order (i.e., curvature continuous).
- \( d = 0 \) and \( b_{11}^2 + b_{12}^2 + b_{22}^2 \neq 0 \): The marching direction vector is defined. Thus, \( p \) and \( q \) are said to intersect tangentially at the neighborhood.

If \( b_{11} \neq 0 \), \( \sigma' = \mathbf{v} = -\frac{a_{21}}{b_{11}} \), and the marching direction is given by,

\[
\mathbf{c}'(s) = \frac{\mathbf{v}}{\sqrt{\mathbf{v} \cdot \mathbf{p}_x}}.
\]

If \( b_{11} = 0 \) and \( b_{22} \neq 0 \), \( \sigma' = \mu = -\frac{a_{11}}{b_{22}} \), then the marching direction is given by,

\[
\mathbf{c}'(s) = \frac{\mathbf{p}_x + \mu \mathbf{p}_x}{\sqrt{\mathbf{p}_x \cdot \mathbf{p}_x}}.
\]

3.2.3. Conventional Solution Methods and Issues

The points of the intersection curves are computed successively by integrating the initial value problem for a system of nonlinear ordinary differential equations (6) using standard numerical techniques such as the Runge-Kutta method, Taylor series method or the Adams-Bashforth method [7]. But when two intersection curves are close to each other, then step size selection becomes complex and incorrect step size may lead to a critical problem, straying or looping [6], which is illustrated in Figure 1 [27].

Figure 2 shows the looping phenomenon when the Runge-Kutta method is used to solve an initial value problem corresponding to Figure 6 where we have two intersecting surfaces. The intersection contains a singular point at \( [0, 0, 0, 0, 0, 0]^T \). With an initial condition \( [\frac{1}{3}, 0, 0, 0, 0, 0]^T \), the system of equations (6) is provided as input to a Matlab ODE solver, ode45, which is based on the Runge-Kutta method and adopts an adaptive step size control scheme. As Figure 2 shows, the Matlab ODE solver breaks down near the singular point.
4. ROBUST MARCHING METHOD

In order to avoid the problems inherent to the conventional numerical methods, we have to rely on a different concept to solve the initial value problem of an ODE system. To ensure robustness in finding roots of the ODE system, researchers have focused on validated schemes using interval arithmetic [26]. The validated ODE solution scheme traces a solution after verifying the existence and uniqueness of the solution at every step. This idea is formulated and implemented in various forms by Moore [26], Krückeberg [9], Eijgenraam [5], Löhner [20] and Nedialkov [28]. After validation, a bound is computed which encloses errors in initial values, truncation errors and round off errors [4].

4.1. Concept of Validated ODE Solver

A validated ODE solving scheme consists of two phases [28]: Algorithm I and Algorithm II. Algorithm I finds an a priori enclosure and a step size (based on validation) such that the existence and uniqueness within the a priori enclosure for the step size is verified. This validation is achieved by applying Picard-Lindelöf operator and Banach’s fixed point theorem [28]. A few methods for validation have been proposed such as the constant enclosure method [5], the polynomial enclosure method [21] or the Taylor series method [4].

Algorithm II deals with the propagation of the solution, reduction of wrapping and further prediction of a new step size for the next step. Wrapping is defined as undesirable overestimation of a solution set of an iteration or recurrence which occurs if this solution set is replaced by a superset of some simpler structure and this super set is then used to compute the enclosures for the next step which may eventually lead to an exponential growth of overestimation [18]. The control of the wrapping effect is a critical issue in this phase and several methods such as a local coordinate transformation method [26], a parallelepiped method and a QR factorization method [20] have been proposed.

4.2. Application to Surface Intersection Problem

Since the marching scheme requires to solve a system of equations (6), we can use a validated ODE solver by formulating the equations presented in Section 3.2 in interval arithmetic with interval initial conditions [27]. The solver produces an a priori enclosure at a step and a corresponding step size, which form a region, called a priori box, where the existence and uniqueness of the solution is verified. The union of such a priori boxes constructs a continuous bound enclosing the exact solution curve in the parametric space, which can be mapped into the model space to provide a gap-free bound in 3D model space [27]. The intersection of bounds in the model space mapped from each surface may further reduce the bound containing the intersection curve [27]. The result of this process can serve as one of the basic building blocks of interval solids introduced in [13,34].

One prominent advantage of the application of the validated ODE solver to the surface to surface intersection problem is the capability of coping with singular points, straying and looping [27]. When the solver approaches singular points or points where two intersection curves get close to each other, a validation condition in Algorithm I of the validated ODE solver gets violated so that the a priori enclosure as well as the step size is adjusted. This adjustment is repeated iteratively until the validation condition is satisfied, which leads the solver to trace the correct solution [27]. This iteration will resolve straying or looping in tracing an intersection curve. If this iteration continues to make the step size less than a certain minimum value, then the iteration stops and the solver reports a singular point, see [27].

4.3. Examples

Figure 3 shows a torus and a cylinder intersecting. We trace one of the four loops of the curves of intersection. We apply the validated ODE solver and map the error bounds in parametric space to obtain strict bounds in the 3D model space. The maximum relative model space error = 0.0187.
Transversal intersection of two tensor product Bézier patches is depicted in Figure 4. Like the previous example we solve the IVP for ODEs using a validated ODE solver and subsequently obtain the model space error bounds. The Figure 4 shows how the 3D model space error bound converges to the true intersection for small values of the error.

Figure 5 represents the intersection of two tensor product Bézier patches. The patches are positioned in such a way that they are tangential to each other and their curve of intersection is a 3D curve. The surface control points are represented as degenerate intervals and are provided as input to a validated ODE solver. The enclosure containing the curve of intersection is mapped from the parameter space to the 3D model space and we obtain rigorous bounds in the 3D model space, which guarantee to contain the true curve of intersection with a maximum relative model space error of 0.002.

Figure 6 shows an example constructed in such a way that there is a singular point in the surface intersection curve segment. Tracing the surface intersection in this example would involve separately tracing the four intersection curve segments, given appropriate starting points.

Application of a conventional ODE system solver, such as the Runge-Kutta or Adams-Bashforth methods would involve the following pathologies:

1. Specifying a starting point which is approximate would mean that the curve traced would not have the singularity or bifurcation. The B-rep model generated would lose topological information and the result may further cause failure in CAD model processing.
2. Straying or looping near the singular region, which are essentially related to the uncertainty of the solver in taking a specific step.

Ideally given a starting point \([\hat{\sigma}_0, \hat{t}_0, \hat{u}_0, \hat{v}_0] = [1, 0, 0, \frac{1}{2}]^T\), we expect a solver to notify us as it approaches a region close to the singularity. The use of the recommended solvers in Matlab such as *ode45* (an implementation of Runge-Kutta method) and *ode113* (implementation of Adam’s method) would result in behavior as erratic as shown in Figures 2 and 7. We show in Figure 8 the behavior of a validated ODE solver which does not march across the singularity. Thus the intersection is traced by separately tracing all the four intersection curve segments.

Now consider the case when one of the surfaces in Figure 6 is perturbed by a small amount in z-direction such that the intersection curves have different topological configuration. The intersection is now just two separate curve segments, even though they lie very close to each other near the previously singular region. Conventional methods show poor behavior near the region where two intersection curves are very close to each other. This is shown by Figure 9 obtained using the Adams-Bashforth method. Note the inconsistency in topology of the intersection curves obtained from conventional methods. The validated ODE solver uses an adaptive step size strategy, easily resolves this case, and behaves well locally close to the near-singular region as shown by Figure 10.

5. CONCLUSIONS

Investigating the effects of floating point arithmetic on the implementation of intersection algorithms has been...
an important area for basic research during the last decade [31]. Methods to enhance the precision of intersection computation, to monitor numerical error contamination and alternate means of performing arithmetic, not relying on imprecise floating point computation alone, have been explored in some detail. Researchers in surface intersection problems during the last decade have already obtained a good understanding of robustness problems when employing floating point arithmetic and of methods to mitigate these problems based on rounded interval arithmetic [12].

As a result of the deficiencies of the conventional numerical methods, recent research tends to focus on exact methods involving rational arithmetic. Much research remains to be done in bringing such methods to the CAD practice, generalizing the arithmetic to go beyond rational and algebraic numbers (eg. involving transcendental numbers of trigonometric form), and to explore more efficient alternatives that are generally applicable in low and high degree problems alike. A different direction of research involves the use of non-conventional interval methods like a validated ODE solver [27], which considers errors arising in the computation as well as initial conditions. It provides a guaranteed bound which encloses the exact solution, and fits well with the concept of robust interval solid modeling [13, 34].

Extension of current intersection methods applied on rational B-spline surfaces to more general and complex surfaces requires further study. Such surfaces include offset, generalized cylinder, blending and medial surfaces, and surfaces arising from the solution of partial differential equations or via recursion techniques.

ACKNOWLEDGEMENTS
This work was funded in part by the NSF (grants No. DMS-0138098 and CCR-0231511).

6. REFERENCES


