Nonlinear Polynomial Systems: Multiple Roots and their Multiplicities

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Motivation

- **Difficulties in handling roots with high multiplicity**
  - **Performance deterioration**
  - **Lack of robustness in numerical computation**
  - **Round-off errors during floating point arithmetic**

- **Limited research on root multiplicity of a system of equations**
  - **Heuristic approaches are needed for practical purposes.**

Objectives

- **Develop practical algorithms to isolate and compute roots and their multiplicities.**
- **Improve the Interval Projected Polyhedron (IPP) algorithms.**

Multiplicity of Roots

- **Univariate Case**
  - **A root a of *f(x)=0* has multiplicity *k* if**

$$f(a)=f'(a)=\cdots=f^{(k-1)}(a)=0,\ \ and\ \ f^{(k)}(a)\neq 0$$

- **Bivariate Case**
  - **Define**

$$V_f=\{(x,y)\in \mathbf{C}\mid f(x,y)=0\}$$

$$V_g=\{(x,y)\in \mathbf{C}\mid g(x,y)=0\}$$

  - **Suppose that z_0 is the only common point of V_f and V_g lying above x_0. Consider $h(x)=Res_y(f,g)$, the resultant of f,g with respect to y. Then the multiplicity of $z_0=(x_0,y_0)$ as a root of the system is the multiplicity of x_0 as a zero of $h(x)$.**

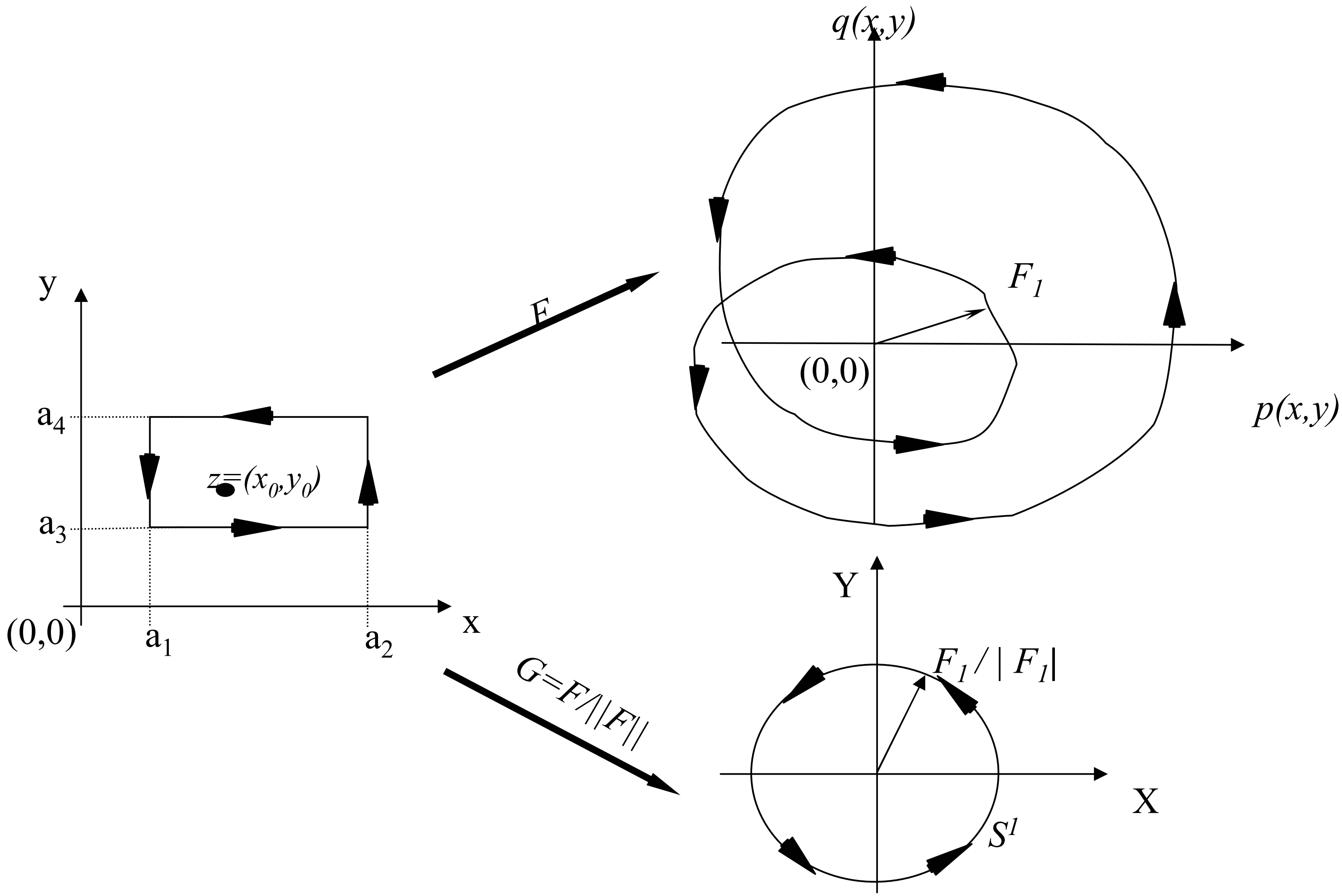
Degree of the Gauss Map

- **Let *p(x,y), q(x,y)* be polynomials with rational coefficients without common factors, of degrees n_1 *and* n_2, and let *F=(p, q).***

- **Let *A* be a rectangle in the plane defined by** $a_1 \le x \le a_2,\ a_3 \le y \le a_4,$ $a_1 < a_2,\ a_3 < a_4,\ a_i \in \mathbf{Q},\ i = 1,2,3,4$ **so that no zero of *F* lies its boundary** ∂A**, and** $p \cdot q$ **does not vanish at its vertices.**
  - Gauss map $G: \partial A \to S^1,\ G = F / \|F\|,$ where S^1 is the unit circle.
  - *G* is continuous ($\|F\| \ne 0 \ \ on \ \ \partial A$).
  - ∂A and S^1 carry the counterclockwise orientation.

- **Degree *d* of *G* : an integer indicating how many times** ∂A **is wrapped around** S^1 **by *G*.**

Illustration of the Gauss Map



The Cauchy Index

- **Preliminaries**
  - *R(x)* : a rational function *q(x)/p(x),* where *p, q* are polynomials.
  - [*a,b*] : a closed interval, *a* < *b*. *R* does not become infinite at the end points.

- **Definition of the *Cauchy index***

  By the *Cauchy index*, $I_a^b R$ of *R* over [*a,b*]*,* we mean $I_a^b R = N_-^+ - N_+^-$ where $N_-^+ (N_+^-)$ denotes the number of points in (*a,b*) at which *R(x)* jumps from $-\infty \; to + \infty (+\infty \; to \; -\infty)$, respectively, as *x* is moving from *a* to *b*. Notice that $I_a^b R = -I_b^a R$ from the definition.

The Cauchy Index (continued)

- **Preliminaries**
  - *A* : a rectangle defined by $[a_1, a_2] \times [a_3, a_4]$ which encloses a zero.
  - *F = (p,q)* does not vanish on the boundary of *A*, ∂A.
  - $p \cdot q$ is not zero at each vertex of *A*.
  - Let

$$R_1 = \frac{q(a_1, y)}{p(a_1, y)},\ R_2 = \frac{q(a_2, y)}{p(a_2, y)},\ R_3 = \frac{q(x, a_3)}{p(x, a_3)},\ R_4 = \frac{q(x, a_4)}{p(x, a_4)}.$$

    Then, we set (for counterclockwise traversal of ∂A)

$$I_A F = I_{a_4}^{a_3} R_1 + I_{a_3}^{a_4} R_2 + I_{a_1}^{a_2} R_3 + I_{a_2}^{a_1} R_4.$$

- **Proposition***

•T. Sakkalis, "*The Euclidean Algorithm and the Degree of the Gauss Map*", SIAM J. Computing. Vol. 19, No. 3, 1990.

$I_A F$ is an even integer and the multiplicity $d = -\frac{1}{2} I_A F.$

Illustrative Example for Multiplicity Computation Using the Cauchy Index

- $p(x) = (x-1/2)^5 = 0$
- A root of $p(x)$, $[a] = [0.49, 0.51]$.
- $P(z)$; $(z = x+iy)$

$$p(z) = (x + iy - \frac{1}{2})^5 = f(x,y) + ig(x,y)$$

- Create

$$A = [0.49, 0.5\ 1] \times [-0.01, 0.01],\quad a_1 = 0.49,\ a_2 = 0.51,\ a_3 = -0.01,\ a_4 = 0.01$$

- Calculate the Cauchy index
  - Roots of $f(x, a_3) = 0$
  - Calculation of

$$I_{a_1}^{a_2} R_3 = -3$$

No.	Roots of $f(x,a_3) = 0$ in [0,1] (from the IPP)
1	[0.46922316412099, 0.46922316512099]
2	[0.49273457408967, 0.492734576204823]
3	[0.499999997363532, 0.500000001889623]
4	[0.507265424645288, 0.507265426808589]
5	[0.530776834861365, 0.530776835861365]

- Roots No. 2, 3, and 4 are selected since they lie within the interval $[a]$.

Illustrative Example (Continued)

- Similarly, $I_{a_3}^{a_4} R_2 = -2,\ I_{a_2}^{a_1} R_4 = 3,\ I_{a_4}^{a_3} R_1 = 2$

- Calculate $I_A F = I_{a_4}^{a_3} R_1 + I_{a_3}^{a_4} R_2 + I_{a_1}^{a_2} R_3 + I_{a_2}^{a_1} R_4 = -10$

- The multiplicity *m* of the root is $d = -\frac{1}{2} I_A F = 5$

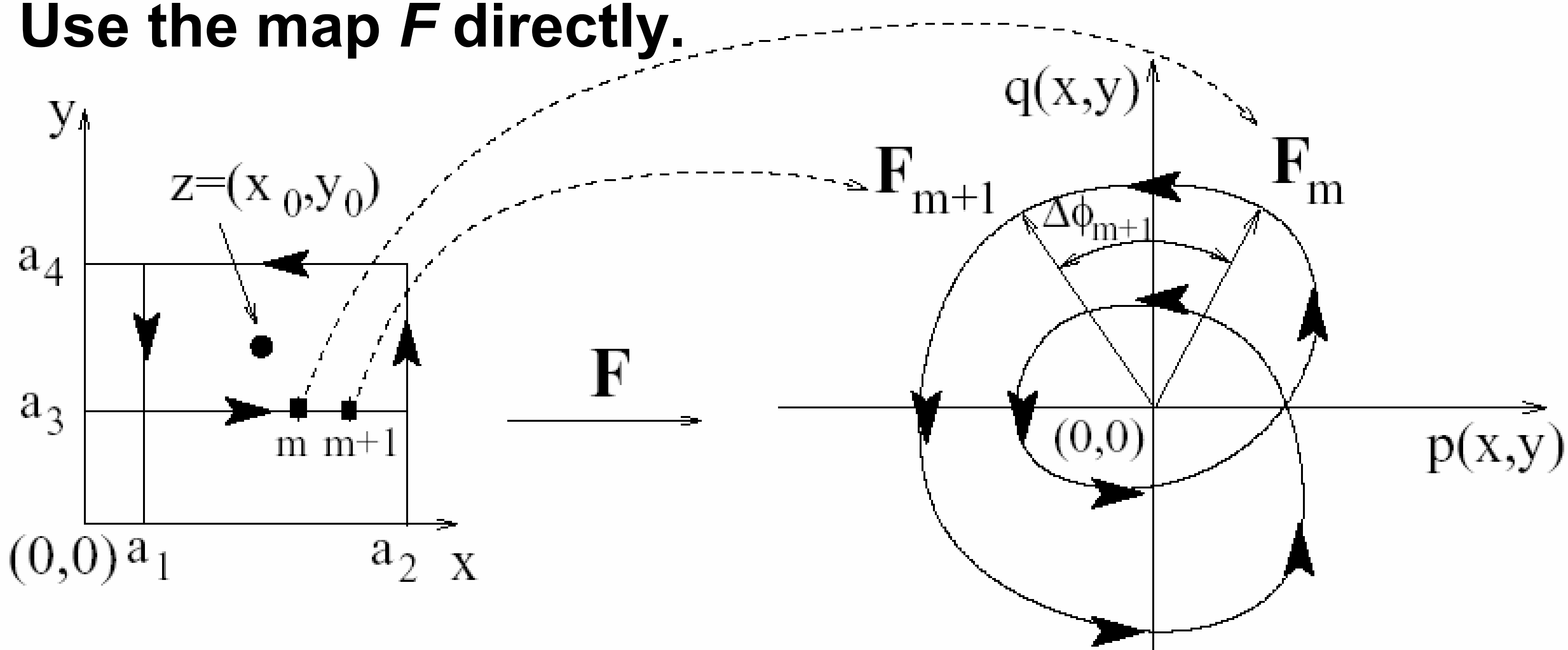
Note

– $I_a^b R = -I_b^a R.$

– *Counterclockwise orientation of* ∂A *is assumed.*

Direct Computation Method

- **Use the map *F* directly.**

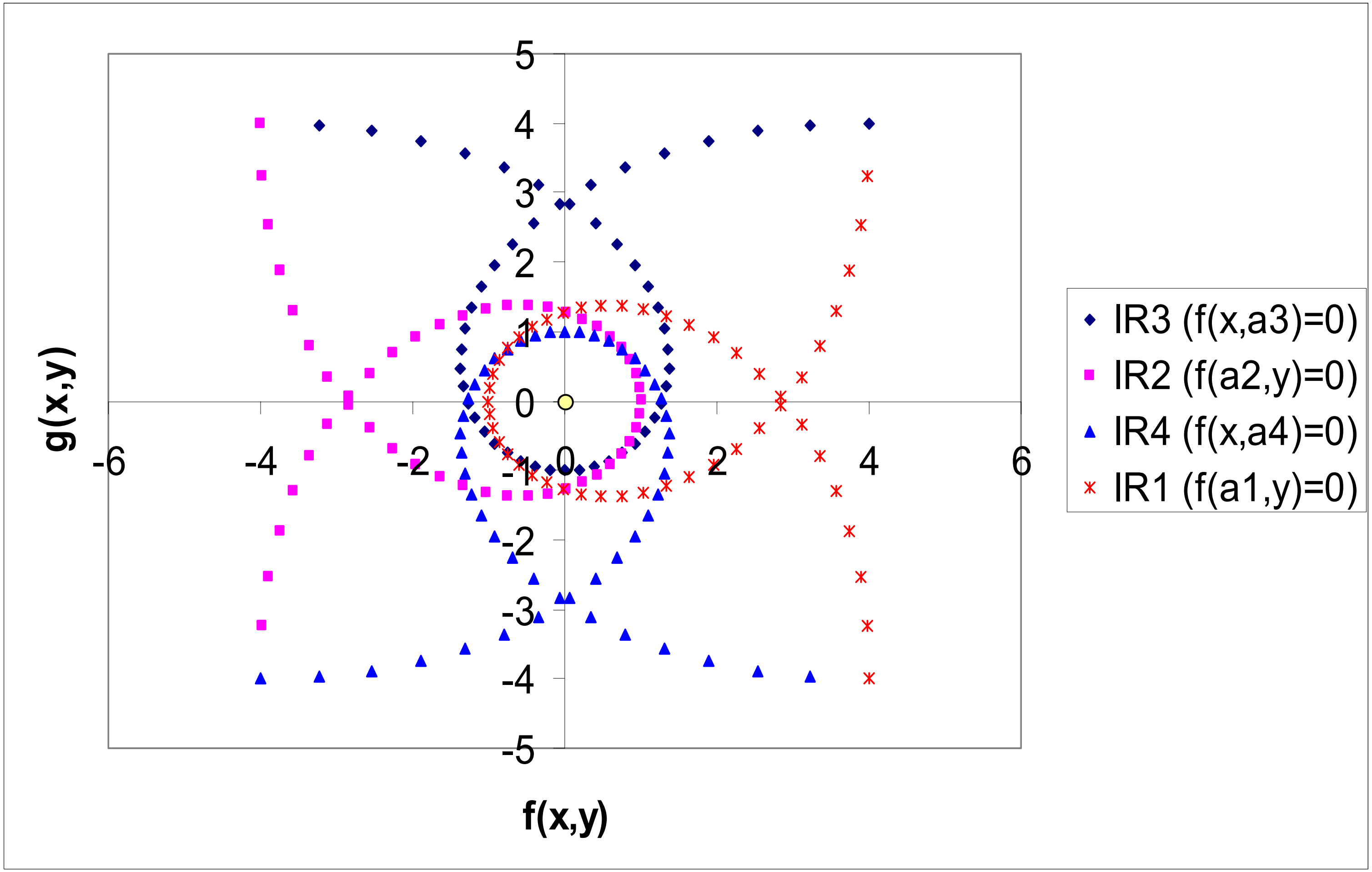


$$\phi_{total} = \sum_{i=0}^{n} \Delta\phi_{i+1}$$

$$d = \frac{\phi_{total}}{2\pi}$$

Direct Computation Method

$$F:\mathbf{R}^2 \to \mathbf{R}^2,\ \ F(x,y)=(f(x,y),g(x,y)).\ \ G:\partial A \to S^1,\ G = F/\|F\|,$$

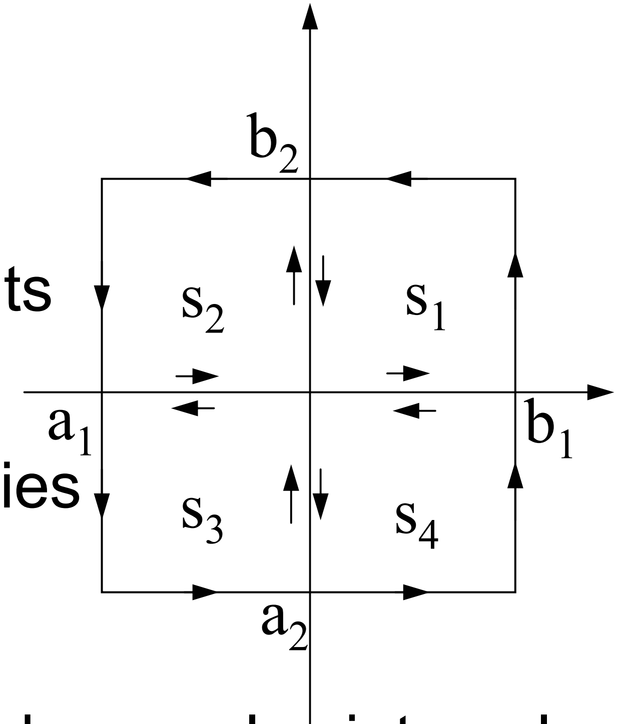


Bisection Algorithm for Solving Univariate Polynomial Equations

- **Univariate polynomial in complex variable *z*. *(Substitute x with a complex variable z = x+iy)***

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z^1 + a_0 = 0$$

- **Input :**
  - initial domain : $S = [a_1, b_1] \times [a_2, b_2]$
  - a complex polynomial : *p(z)*
  - tolerance, number of sample points
- **Output**
  - real and complex roots, multiplicities
- **Algorithm**
  - Quadtree decomposition
  - Direct degree computation method : complex interval arithmetic.



Examples

- Wilkinson polynomial

$$p(t) = \prod_{i=1}^{20} \left(t - \frac{i}{20} \right)$$

No.	Multiplicity	Roots
1	1	[0.05,0.05]+i[-5.769e-10,5.769e-10]
2	1	[0.1,0.1]+i[-5.866e-10,5.866e-10]
3	1	[0.15,0.15]+i[-5.947e-10,5.947e-10]
4	1	[0.2,0.2]+i[-5.947e-10,5.947e-10]
5	1	[0.25,0.25]+i[-5.898e-10,5.898e-10]
6	1	[0.3,0.3]+i[-5.792e-10,5.792e-10]
7	1	[0.35,0.35]+i[-5.792e-10,5.792e-10]
8	1	[0.4,0.4]+i[-5.792e-10,5.792e-10]
9	1	[0.45,0.45]+i[-5.792e-10,5.792e-10]
10	1	[0.5,0.5]+i[-5.745e-10,5.745e-10]
11	1	[0.55,0.55]+i[-5.745e-10,5.745e-10]
12	1	[0.6,0.6]+i[-5.745e-10,5.745e-10]
13	1	[0.65,0.65]+i[-5.745e-10,5.745e-10]
14	1	[0.7,0.7]+i[-5.745e-10,5.745e-10]
15	1	[0.75,0.75]+i[-5.745e-10,5.745e-10]
16	1	[0.8,0.8]+i[-5.745e-10,5.745e-10]
17	1	[0.85,0.85]+i[-5.745e-10,5.745e-10]
18	1	[0.9,0.9]+i[-5.745e-10,5.745e-10]
19	1	[0.95,0.95]+i[-5.745e-10,5.745e-10]
20	1	[1,1]+i[-5.747e-10,5.747e-10]

- Complicated Polynomial (degree 22)

$$p(t) = (t^2 + t + 1)^2 (t-1)^4 (t^3 + t^2 + t + 1)^3 (t-2)(t-4)^4$$

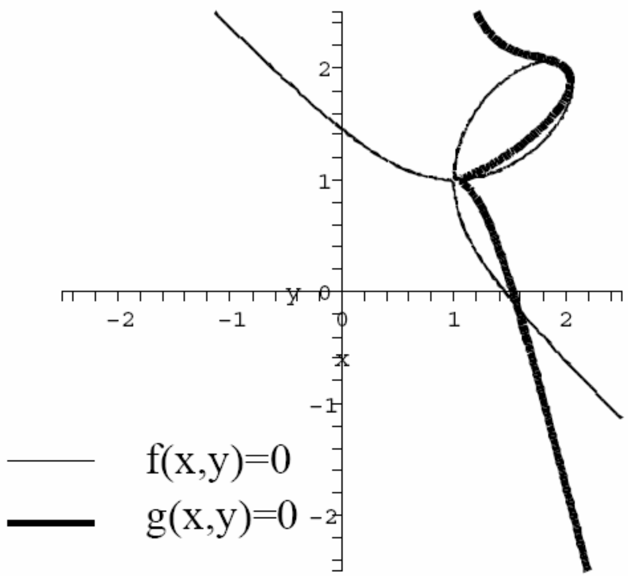
No.	Multiplicity	Roots
1	3	[-5.956e-10,5.956e-10]+i[1,1]
2	4	[1,1]+i[-5.956e-10,5.956e-10]
3	4	[4,4]+i[-5.939e-10,5.939e-10]
4	2	[-0.5,-0.5]+i[0.866,0.866]
5	3	[-1,-1]+i[-5.956e-10,5.956e-10]
6	2	[-0.5,-0.5]+i[-0.866,-0.866]
7	3	[-5.956e-10,5.956e-10]+i[-1,-1]
8	1	[2,2]+i[-5.94e-10,0]

Solving a Bivariate Polynomial System

- **Change of Coordinates**
  - CR : f and g are regular in y.
  - CU : whenever two points (x_0,y_0) and (x_1,y_1) satisfy $f=g=0$, then $y_0=y_1$.
- **Solving a Bivariate Polynomial System**
  - Let f,g satisfy CR and CU and let $h(x)=Res_y(f,g)$. Then the roots of the system $f=g=0$ are in a one to one correspondence with the roots of $h(x)$. Moreover, $z_i=(x_i,y_i)$ is a real root if and only if x_i is a real root of $h(x)$.
  - Let $h(x)=Res_y(f,g)$ and $l(y)=Res_x(f,g)$ and $a_{ij}=[t_i,t_{i+1}]x[s_j,s_{j+1}]$ where in each subinterval $[t_i,t_{i+1}]$ or $[s_j,s_{j+1}]$ there exist precisely one root of $h(x)$ and $l(y)$, *respectively*. If a_{ij} encloses a real root of $f=g=0$, then the following must be true

$$0\in f([t_i,t_{i+1}],[s_j,s_{j+1}])\times g([t_i,t_{i+1}],[s_j,s_{j+1}])$$

Solving a Bivariate Polynomial System : Example



$$\begin{aligned} f(x,y) &= x^3 - 3x^2 + 5x - 4 + y^3 \\ &\quad -3y^2 + 5y - 2xy = 0, \\ g(x,y) &= 2x^3 - 2x^2 + x - 4 - 4x^2y + 2xy \\ &\quad +9y + 3xy^2 - 8y^2 + y^3 = 0, \end{aligned}$$

$$\begin{aligned} h(x) &= 56x^9 - 704x^8 + 3880x^7 - 12304x^6 \\ &\quad +24744x^5 - 32736x^4 + 28504x^3 \\ &\quad -15760x^2 + 5024x - 704. \end{aligned}$$

$$\begin{aligned} l(y) &= -56y^9 + 608y^8 - 2824y^7 + 7312y^6 \\ &\quad -11496y^5 + 11136y^4 - 6328y^3 \\ &\quad +1744y^2 - 32y - 64. \end{aligned}$$

Root (x,y)	d
[0.999999978, 1.00000001]x[0.99999994, 1.00000001]	5
[1.57142855, 1.57142859]x[-0.142857209, -0.142857134]	1
[1.99999999, 2.00000003]x[1.99999996, 2.00000003]	3

Conclusions

- **Study of the topological degree and multiple roots of univariate and bivariate polynomial systems in the context of geometric modeling.**
- **Development of practical algorithms for isolating and computing multiple roots of univariate and bivariate polynomial systems.**
- **Basis for further research needed in addressing the general problem of single and multiple roots of nonlinear polynomial systems in *n* variables.**