Shape Interrogation II

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Polynomials are used in various branches of computational science.

They can be found in mathematics, computer science, engineering and many other fields.

There are two basic reasons for that:
- Most functions can be approximated by polynomial functions, and
- They are rather easy to use in a computer code.

Thus, they serve as good substitutes for functions that are difficult to deal with.
In this talk we will discuss some of their applications in Computer Aided Geometric Design and Geometric Modeling.

In particular, we will discuss:

- Polynomial systems and their solutions
- Elements of elimination theory
- Polynomial maps
- Some Problems of this Area.
A Strange Example

- As an indication of the difference in moving from one dimension to the next, even for simple functions—like polynomials—let us consider the following:

- Example 1. Every polynomial function $y = p(x)$ with $p(x) > 0$, $\forall x \in \mathbb{R}$ has at least one (real) critical point.

$$\lim_{|x| \to \infty} p(x) = \infty.$$ 

- Example 2. The polynomial

$$p(x, y) = (x^2 y - x - 1)^2 + x^2$$

- has the property that, for every $(x, y) \in \mathbb{R}^2$, $p(x, y) > 0$,
- but the function $p(x, y)$ does not have any (real) critical point.

$$\lim_{|(x, y)| \to \infty} p(x, y) \text{ Does not exist.}$$
$y = p(x)$
Polynomial Systems

- Polynomials are popular in curve and surface representation.
- Thus, many critical problems in CAGD, such as surface interrogation, are reduced to finding the zero set of a system of polynomial equations

\[ f(x) = 0 \]

where \( f = (f_1, \ldots, f_n) \) and each \( f_i \) is a polynomial of \( m \) independent variables \( x = (x_1, \ldots, x_m) \).
Polynomial Systems

• Several root-finding methods for polynomial systems have been used in practice.

• These can be categorized as:
  – Algebraic and hybrid methods,
  – Homotopy methods, and
  – Subdivision methods.

• Among those types, the subdivision methods have been widely used in practice.

• The Interval Projected Polyhedral (IPP) algorithm is one example, and it has successfully been applied to various problems.
Motivation

• Difficulties in handling roots with high multiplicity
  - Performance deterioration
  - Lack of robustness in numerical computation
  - Round-off errors during floating point arithmetic

• Limited research on root multiplicity of a system of equations
  - Heuristic approaches are needed for practical purposes.
Objectives

• Develop practical algorithms to isolate and compute roots and their multiplicities.

• Improve the Interval Projected Polyhedron (IPP) algorithms.
**Multiplicity of Roots**

- **Univariate Case**
  - A root $a$ of $f(x)=0$ has multiplicity $k$ if
  
  $$f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0, \text{ and } f^{(k)}(a) \neq 0$$

- **Bivariate Case**
  - Define
    
    $$V_f = \{(x, y) \in \mathbb{C} \mid f(x, y) = 0\}$$
    
    $$V_g = \{(x, y) \in \mathbb{C} \mid g(x, y) = 0\}$$

  - Suppose that $z_0$ is the only common point of $V_f$ and $V_g$ lying above $x_0$. Consider $h(x)=\text{Res}_y(f,g)$, the resultant of $f,g$ with respect to $y$. Then the multiplicity of $z_0=(x_0,y_0)$ as a root of the system is the multiplicity of $x_0$ as a zero of $h(x)$.  

  \[\text{Slide No. 11}\]
Degree of the Gauss Map

- Let $p(x,y), q(x,y)$ be polynomials with rational coefficients without common factors, of degrees $n_1$ and $n_2$, and let $F=(p, q)$.

- Let $A$ be a rectangle in the plane defined by $a_1 \leq x \leq a_2, \ a_3 \leq y \leq a_4,$
  $a_1 < a_2, \ a_3 < a_4, \ a_i \in \mathbb{Q}, \ i=1,2,3,4$ so that no zero of $F$ lies its boundary $\partial A$, and $p \cdot q$ does not vanish at its vertices.
  - Gauss map $G: \partial A \rightarrow S^1, \ G = F / \| F \|$, where $S^1$ is the unit circle.
  - $G$ is continuous ($\| F \| \neq 0$ on $\partial A$).
  - $\partial A$ and $S^1$ carry the counterclockwise orientation.

- Degree $d$ of $G$: an integer indicating how many times $\partial A$ is wrapped around $S^1$ by $G$.
Illustration of the Gauss Map

\[ S \]

\[ G = F / \|F\| \]

\[ F \]

\[ (0,0) \]

\[ p(x,y) \]

\[ q(x,y) \]

\[ F_1 \]

\[ S^l \]

\[ F_1 / |F_1| \]
The Cauchy Index

• Preliminaries
  – $R(x)$: a rational function $q(x)/p(x)$, where $p$, $q$ are polynomials.
  – $[a,b]$ : a closed interval, $a < b$. $R$ does not become infinite at the end points.

• Definition of the Cauchy index
  By the Cauchy index, $I_a^b R$ of $R$ over $[a,b]$, we mean $I_a^b R = N_+^+ - N_+^-$
  where $N_+^+ (N_+^-)$ denotes the number of points in $(a,b)$ at which $R(x)$ jumps from $-\infty$ to $+\infty$ ($+\infty$ to $-\infty$), respectively, as $x$ is moving from $a$ to $b$. Notice that $I_a^b R = -I_b^a R$ from the definition.
The Cauchy Index (continued)

• Preliminaries
  – \( A \): a rectangle defined by \([ a_1, a_2] \times [ a_3, a_4] \) which encloses a zero.
  – \( F = (p,q) \) does not vanish on the boundary of \( A \), \( \partial A \).
  – \( p \cdot q \) is not zero at each vertex of \( A \).
  – Let \( R_1 = \frac{q(a_1, y)}{p(a_1, y)}, R_2 = \frac{q(a_2, y)}{p(a_2, y)}, R_3 = \frac{q(x, a_3)}{p(x, a_3)}, R_4 = \frac{q(x, a_4)}{p(x, a_4)} \).

Then, we set (for counterclockwise traversal of \( \partial A \) )

\[
I_A F = I_{a_1} a_3 R_1 + I_{a_2} a_4 R_2 + I_{a_4} a_3 R_3 + I_{a_2} a_1 R_4.
\]

• Proposition* •T. Sakkalis, “The Euclidean Algorithm and the Degree of the Gauss Map”,

\( I_A F \) is an even integer and the multiplicity \( d = -\frac{1}{2} I_A F \).
Illustrative Example for Multiplicity Computation Using the Cauchy Index

- \( p(x) = (x-\frac{1}{2})^5 = 0 \)
- A root of \( p(x) \), \([ a ] = [0.49,0.51] \).
- \( P(z); (z = x+iy) \)
  \[
  p(z) = (x + iy - \frac{1}{2})^5 = f(x, y) + ig(x, y)
  \]
- Create
  \[
  A = [0.49,0.51] \times [-0.01,0.01], \quad a_1 = 0.49, \quad a_2 = 0.51, \quad a_3 = -0.01, \quad a_4 = 0.01
  \]
- Calculate the Cauchy index
  - Roots of \( f(x, a_3) = 0 \)
  - Calculation of
    \[
    I_{a_1}^a_2 R_3 = -3
    \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Roots of ( f(x,a_3) = 0 ) in [0,1] (from the IPP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.46922316412099, 0.46922316512099]</td>
</tr>
<tr>
<td>2</td>
<td>[0.49273457408967, 0.492734576204823]</td>
</tr>
<tr>
<td>3</td>
<td>[0.499999997363532, 0.500000001889623]</td>
</tr>
<tr>
<td>4</td>
<td>[0.507265424645288, 0.507265426808589]</td>
</tr>
<tr>
<td>5</td>
<td>[0.530776834861365, 0.530776835861365]</td>
</tr>
</tbody>
</table>

- Roots No. 2, 3, and 4 are selected since they lie within the interval \([ a ]\).
Similarly, \[ I_{a_3} R_2 = -2, \quad I_{a_1} R_4 = 3, \quad I_{a_4} R_1 = 2 \]

Calculate \[ I_A F = I_{a_4} R_1 + I_{a_3} R_2 + I_{a_1} R_3 + I_{a_2} R_4 = -10 \]

The multiplicity \( m \) of the root is \[ d = -\frac{1}{2} I_A F = 5 \]

**Note**
- \( I_b^b R = -I_b^a R \).
- **Counterclockwise** orientation of \( \partial A \) is assumed.
Direct Computation Method

- Use the map $F$ directly.

$$\phi_{total} = \sum_{i=0}^{n} \Delta \phi_{i+1}$$

$$d = \frac{\phi_{total}}{2\pi}$$
Direct Computation Method

\( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = (f(x, y), g(x, y)) \). \( G : \partial A \rightarrow S^1, \quad G = F / \|F\| \),

\begin{align*}
\text{IR}_1 (f(a_1, y) = 0) \\
\text{IR}_2 (f(a_2, y) = 0) \\
\text{IR}_3 (f(x, a_3) = 0) \\
\text{IR}_4 (f(x, a_4) = 0) \\
\end{align*}
Bisection Algorithm for Solving Univariate Polynomial Equations

• Univariate polynomial in complex variable $z$. 
(Substitute $x$ with a complex variable $z = x + iy$)

$$ p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 $$

• Input:
  – initial domain: $S = [a_1, b_1] \times [a_2, b_2]$
  – a complex polynomial: $p(z)$
  – tolerance, number of sample points

• Output
  – real and complex roots, multiplicities

• Algorithm
  – Quadtree decomposition
  – Direct degree computation method: complex interval arithmetic.
Examples

- Wilkinson polynomial

\[ p(t) = \prod_{i=1}^{20} \left( t - \frac{i}{20} \right) \]

- Complicated Polynomial (degree 22)

\[ p(t) = (t^2 + t + 1)^2(t - 1)^4(t^3 + t^2 + t + 1)^3(t - 2)(t - 4)^4 \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Multiplicity</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>([0.05,0.05] + i[-5.769e-10,5.769e-10])</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>([0.1,0.1] + i[-5.866e-10,5.866e-10])</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>([0.15,0.15] + i[-5.947e-10,5.947e-10])</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>([0.2,0.2] + i[-5.947e-10,5.947e-10])</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>([0.25,0.25] + i[-5.898e-10,5.898e-10])</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>([0.3,0.3] + i[-5.792e-10,5.792e-10])</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>([0.35,0.35] + i[-5.792e-10,5.792e-10])</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>([0.4,0.4] + i[-5.792e-10,5.792e-10])</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>([0.45,0.45] + i[-5.792e-10,5.792e-10])</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>([0.5,0.5] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>([0.55,0.55] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>([0.6,0.6] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>([0.65,0.65] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>([0.7,0.7] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>([0.75,0.75] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>([0.8,0.8] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>([0.85,0.85] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>([0.9,0.9] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>([0.95,0.95] + i[-5.745e-10,5.745e-10])</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>([1,1] + i[-5.745e-10,5.745e-10])</td>
</tr>
</tbody>
</table>
Solving a Bivariate Polynomial System

• Change of Coordinates
  - CR : $f$ and $g$ are regular in $y$.
  - CU : whenever two points $(x_0,y_0)$ and $(x_1,y_1)$ satisfy $f=g=0$, then $y_0=y_1$.

• Solving a Bivariate Polynomial System
  - Let $f,g$ satisfy CR and CU and let $h(x)=\text{Res}_y(f,g)$. Then the roots of the system $f=g=0$ are in a one to one correspondence with the roots of $h(x)$. Moreover, $z_i=(x_i,y_i)$ is a real root if and only if $x_i$ is a real root of $h(x)$.
  - Let $h(x)=\text{Res}_y(f,g)$ and $l(y)=\text{Res}_x(f,g)$ and $a_{ij}=[t_i,t_{i+1}][s_j,s_{j+1}]$ where in each subinterval $[t_i,t_{i+1}]$ or $[s_j,s_{j+1}]$ there exist precisely one root of $h(x)$ and $l(y)$, respectively. If $a_{ij}$ encloses a real root of $f=g=0$, then the following must be true:
    \[0 \in f([t_i,t_{i+1}],[s_j,s_{j+1}]) \times g([t_i,t_{i+1}],[s_j,s_{j+1}])\]
Solving a Bivariate Polynomial System : Example

\[ f(x, y) = x^3 - 3x^2 + 5x - 4 + y^3 - 3y^2 + 5y - 2xy = 0, \]
\[ g(x, y) = 2x^3 - 2x^2 + x - 4 - 4x^2y + 2xy + 9y + 3xy^2 - 8y^2 + y^3 = 0, \]
\[ h(x) = 56x^9 - 704x^8 + 3880x^7 - 12304x^6 + 24744x^5 - 32736x^4 + 28504x^3 - 15760x^2 + 5024x - 704. \]
\[ l(y) = -56y^9 + 608y^8 - 2824y^7 + 7312y^6 - 11496y^5 + 11136y^4 - 6328y^3 + 1744y^2 - 32y - 64. \]

<table>
<thead>
<tr>
<th>Root (x,y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.999999978, 1.000000001] x [0.999999994, 1.000000001]</td>
</tr>
<tr>
<td>[1.57142855, 1.57142859] x [-0.142857209, -0.142857134]</td>
</tr>
<tr>
<td>[1.99999999, 2.000000003] x [1.999999996, 2.000000003]</td>
</tr>
</tbody>
</table>
Elimination Theory

I. Resultants
   • Sylvester Resultant
   • Macaulay Resultant
   • Sparse Resultant
   • D-Resultant

II. Groebner Bases

III. Symbolic System Solving
Elements of Resultant Theory

- Let:
  \[ a(t) = a_n t^n + \cdots + a_1 t + a_0 \]
  \[ b(t) = b_m t^m + \cdots + b_1 t + b_0 \]
- non zero polynomials, with complex coefficients.
- The resultant of \( a, b \) wrt \( t \) (or the \( t \)-resultant), \( \text{Res}_t(a, b) = R \) is

\[
R = \begin{vmatrix}
    a_n & a_{n-1} & \cdots & a_0 \\
    a_n & \cdots & a_1 & a_0 \\
    \vdots & & \ddots & \vdots \\
    a_n & \cdots & \cdots & a_0 \\
    b_m & \cdots & \cdots & b_0 \\
    \vdots & & & \ddots \\
    b_m & \cdots & \cdots & b_0 
\end{vmatrix}
\]
- Observe that \( \text{Res}_t(a, b) \in \mathbb{C} \).
Properties of the Resultant

• Let us see some well known properties of the resultant:

• **Property 1.** There exist polynomials $A(t), B(t) \in \mathbb{C}[t]$ of degrees respectively, $n', m', m < n$ so that

\[
a(t)A(t) + b(t)B(t) = \text{Res}_t(a, b).
\]

• **Property 2.** $\text{Res}_t(a, b) = 0 \iff a(t)$ and $b(t)$ have a common factor of positive degree.

• **Property 3.**
  
  Let, \( a(x, y) = a_n y^n + a_{n-1}(x) y^{n-1} + \cdots + a_0(x) \)

  \[
  b(x, y) = \sum_{i=0}^{m} b_{m-i}(x) y^{m-i} \in k[y][x]
  \]

  with $a_n$ or $b_m \in \mathbb{C}^*$, and consider $p(x) = \text{Res}_y(a, b)$. If $x_0$ is a root of $p(y)$, then there exists $y_0 \in \mathbb{C}$ with the property $a(x_0, y_0) = b(x_0, y_0) = 0$.
Cramer’s Rule

• Let $f(x,y), g(x,y) \in C[x,y]$ two nonconstant polynomials, and let $x, y$ be indeterminates.

• Consider

\[ F : C^2 \rightarrow C^2, \quad F = (f, g) \]

\[ A(x,u,v) = \text{Res}_y (f - u, g - v), \]

\[ B(y,u,v) = \text{Res}_x (f - u, g - v) \]

• with $F(0,0) = (0,0)$. 
Cramer’s Rule

• **Theorem**[Cramer’s Rule] $F$ has a polynomial inverse if and only if:
  
  $$A(x, u, v) = ax + A_0(u, v),$$

  • and

  $$B(y, u, v) = by + B_0(u, v), \text{ with } ab \neq 0$$

• Moreover, if

  $$G(x, y) := \left(-\frac{A_0(x, y)}{a}, -\frac{B_0(x, y)}{b}\right),$$

• Then $G$ is the inverse of $F$.

• In addition,

  $$\deg F = \deg F^{-1}$$