Tracing Surface Intersections with Validated ODE System Solver

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June 2004

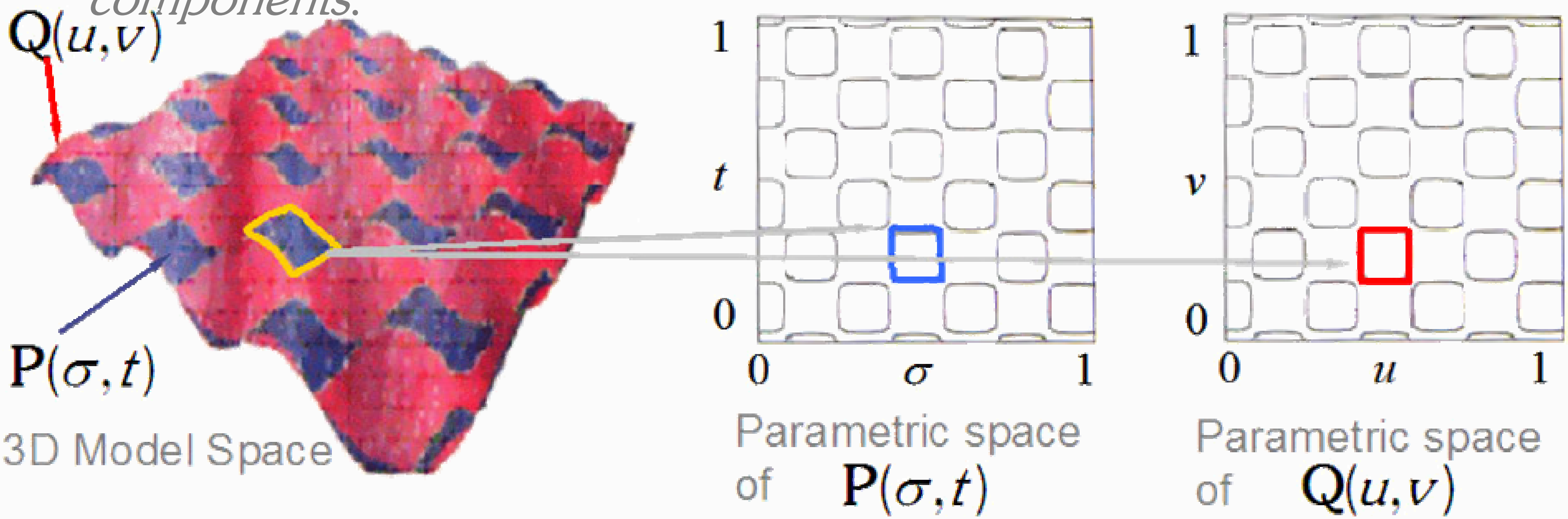
ACM Solid Modeling Conference 2004

Massachusetts Institute of Technology

Introduction

Background

- Intersection of two *parametric surfaces*, $P(\sigma, t) = Q(u, v)$ defined in *parametric spaces* $0 \le \sigma, t \le 1$ and $0 \le u, v \le 1$ can have *multiple components*.



- An *intersection curve segment* is represented by a continuous trajectory in *parametric space.*

GRANDINE T. A., KLEIN F. W.: A new approach to the surface intersection problem. Computer Aided Geometric Design 14, 2 (1997), 111–134.

Problem Formulation
Vector IVP for ODE

- *Transversal* as well as *tangential* intersections can be formulated as a system of *ordinary differential equations* (ODEs) in *parametric space*.

- Given a *starting point (initial condition)* belonging to an *intersection curve segment*, we can integrate the system of ODEs.

- The system of ODEs with the *starting point* represents an initial value problem (IVP).
  - Written in vector notation as:

$$\begin{bmatrix} \frac{d\sigma}{ds} \\ \frac{dt}{ds} \\ \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = \begin{bmatrix} f_1(\sigma,t,u,v) \\ f_2(\sigma,t,u,v) \\ f_3(\sigma,t,u,v) \\ f_4(\sigma,t,u,v) \end{bmatrix}$$

$$\frac{d\mathbf{y}}{ds} = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(\mathbf{0}) = \mathbf{y}_0$$

Error Bounds in Parametric Space

Validated Interval Scheme (Application to SSI)

- For strict bounds for IVPs in *parametric space,* we employ a *validated interval scheme** for ODEs.
- We represent the surfaces as *interval surfaces.*
    - *Interval surfaces* have *interval coefficients* and are written as:

$$[\mathrm{P}](\sigma,t) \text{ and } [\mathrm{Q}](u,v)$$

- We obtain a *vector interval ODE system* :

$$\left[\frac{d\sigma}{ds}\ \frac{dt}{ds}\ \frac{du}{ds}\ \frac{dv}{ds}\right]^T = \frac{d\mathrm{y}}{ds} = \mathrm{f}([\mathrm{y}(s)])$$

- The error in *starting point* is bounded by an *initial interval.*
    - With an *interval initial condition* :

$$[\mathrm{y}_0] = \left[[\sigma_0]\ \ [t_0]\ \ [u_0]\ \ [v_0]\right]^T$$

*NEDIALKOV N. S.: *Computing the Rigorous Bounds on the Solution of an Initial Value Problem for an Ordinary Differential Equation*. PhD thesis, University of Toronto, Toronto, Canada, 1999.

Error Bounds in Parametric Space

Validated Interval Scheme (Overview)

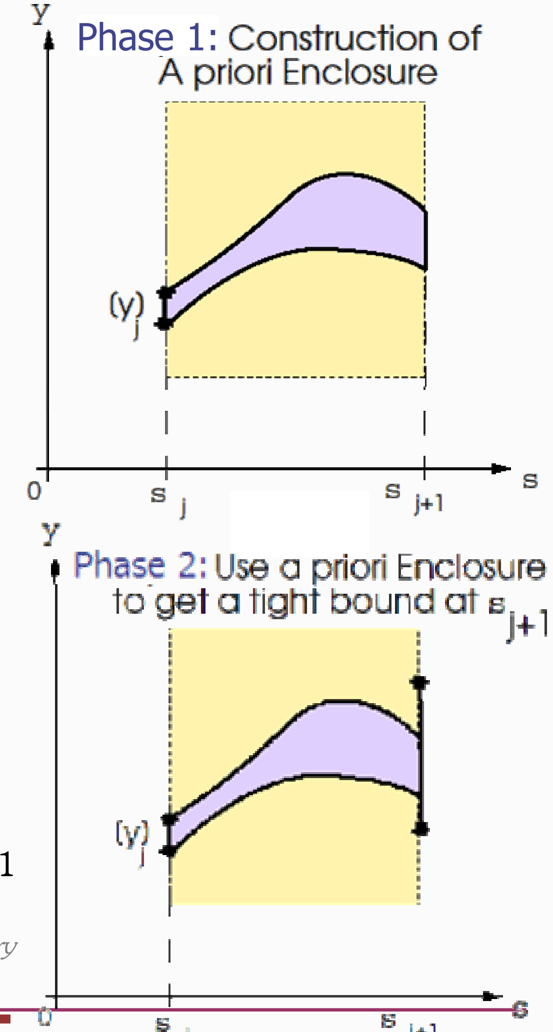
- One *step* of a *validated interval scheme** is done in *two* phases:
  - Phase I Algorithm

    (Verifying the *existence* and *uniqueness*)
    - A *step size* $h_j = s_{j+1} - s_j$
    - An *a priori enclosure* $[\tilde{y}_j]$ such that:

$$y(s) \in [\tilde{y}_j], \quad \forall \ s \in [s_j, s_{j+1}]$$

  - Phase II Algorithm

    (Computing an interval valued function)
    - Using $[\tilde{y}_j]$ compute a *tighter bound* $[y_{j+1}]$ at s_{j+1}



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Validated Error Bounds in 3D

Flow Chart Description

Input
Error bound on the *starting point* in both *parametric spaces*

→

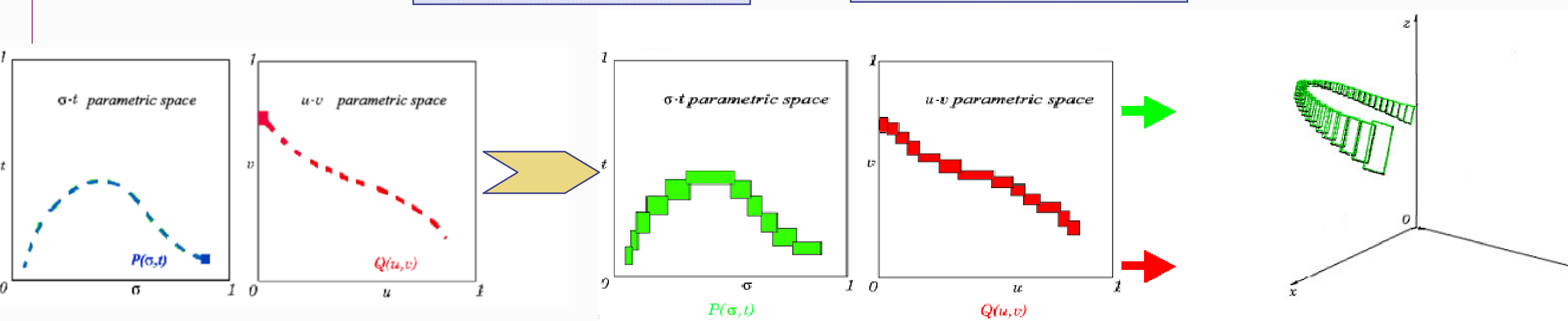
Tracing using validated ODE solvers to obtain Validated Error bounds in parametric space of both surfaces

→

Mapping into 3D model space using real interval arithmetic

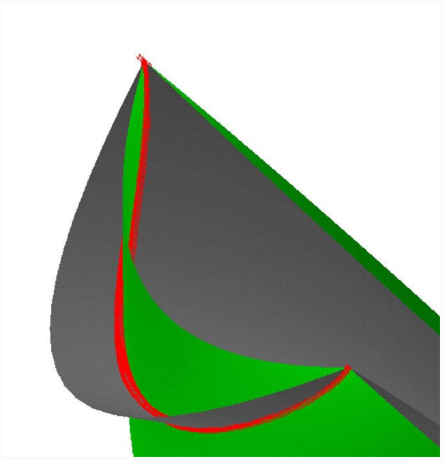
→

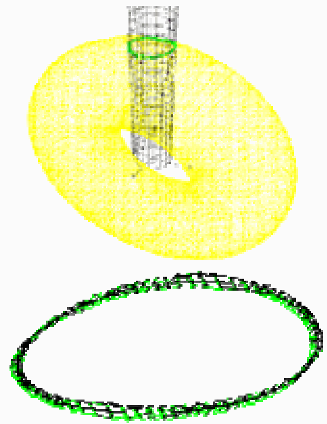
Output
Validated Error bounds in 3D model space

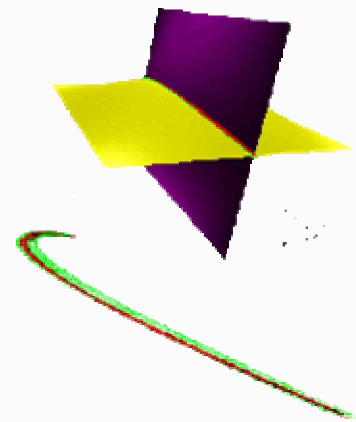


Results & Examples

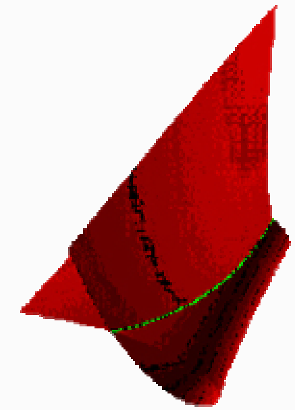
Error Bounds in 3D Model Space (Transversal and Tangential)



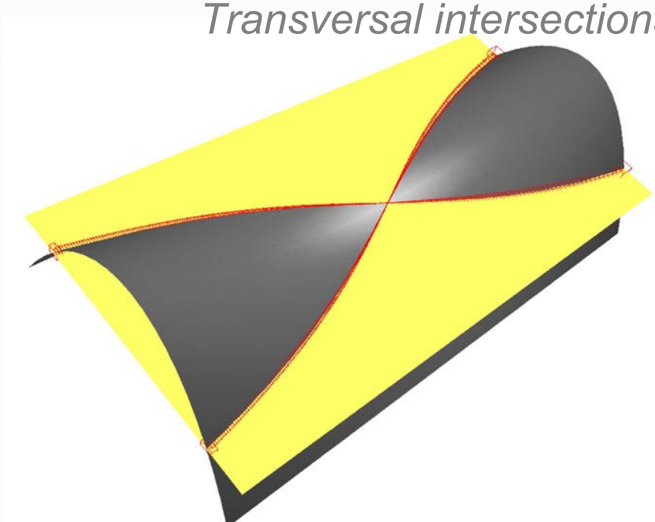




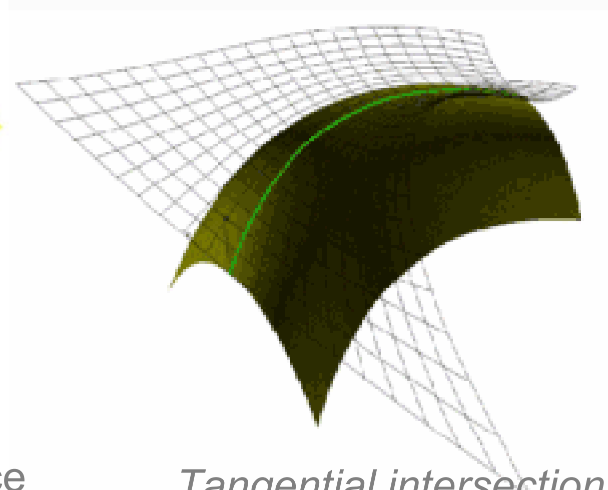
Transversal intersections of parametric surfaces

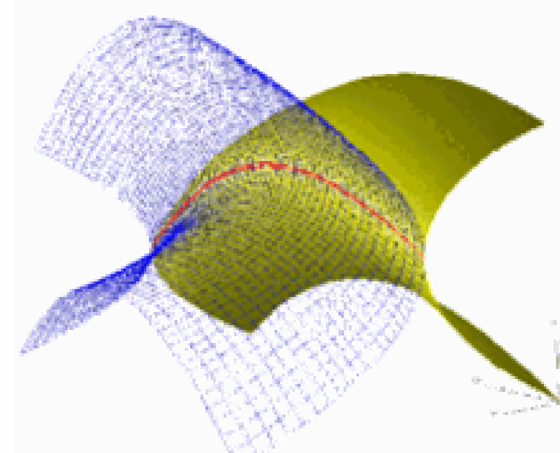


Self intersection of a *bi-cubic* surface



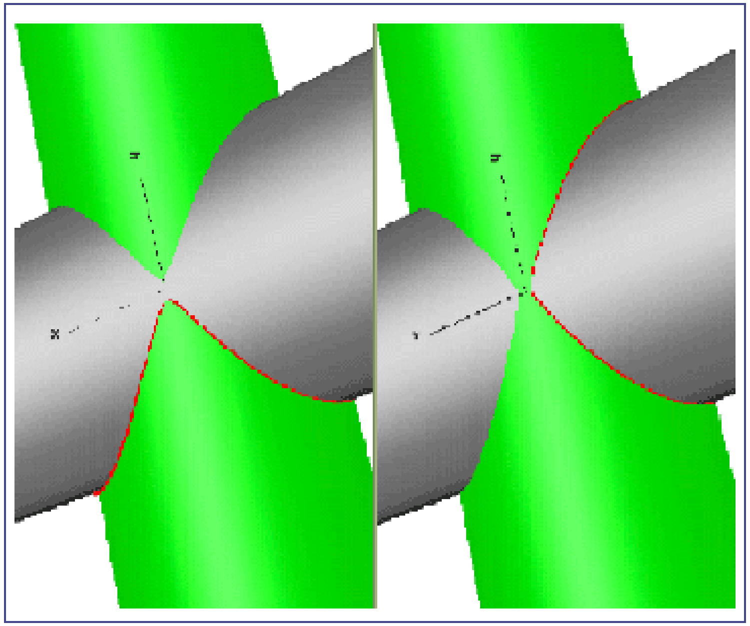
Intersection of a *hyperbolic* surface with a plane

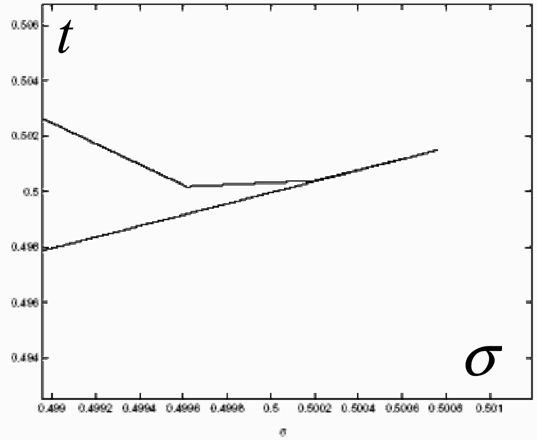




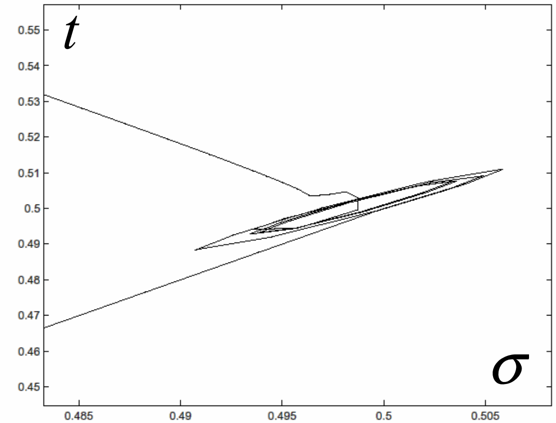
Tangential intersections of parametric surfaces

Results & Examples
Preventing Straying or Looping



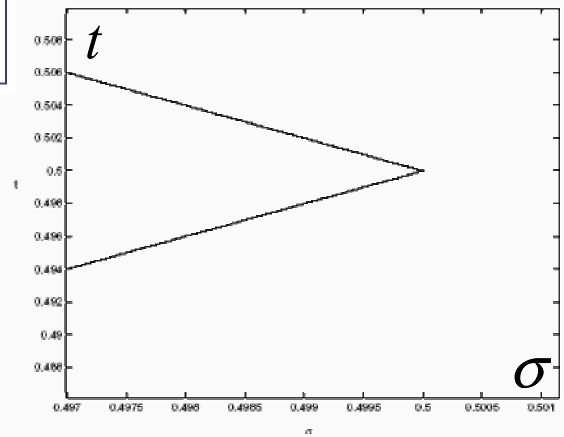


Adams-Bashforth



Runge-Kutta

Perturbation	Steps Required by the Method
+0.000003	1139
0.0	Singularity Reported
-0.000003	1303



Result from a validated interval scheme

Validated ODE solver can correctly trace the *intersection curve segment* even through closely spaced features, where standard methods fail.

Conclusions
Merits

- We realize *validated error bounds* in *3D model space* which enclose the *true curve of intersection.*
- The scheme can prevent the phenomenon of *straying or looping*.
- The scheme can accommodate the errors in:
  - initial condition
  - rounding during digital computation
- Method can accommodate perturbation in the surfaces itself.
- *Validated error bounds* for surface intersection is essential in *interval boundary representation* for consistent *solid models**.

*SAKKALIS T., SHEN G., PATRIKALAKIS N. M.:Topological and geometric properties of interval solid models. *Graphical Models 63*, 3 (2001), 163–175.