Nonlinear Polynomial Systems: Multiple Roots and their Multiplicities

K. H. Ko, T. Sakkalis, N. M. Patrikalakis

Massachusetts Institute of Technology
Motivation

- Difficulties in handling roots with high multiplicity
  - Performance deterioration
  - Lack of robustness in numerical computation
  - Round-off errors during floating point arithmetic

- Limited research on root multiplicity of a system of equations
  - Heuristic approaches are needed for practical purposes.
Objectives

• Develop practical algorithms to isolate and compute roots and their multiplicities.

• Improve the Interval Projected Polyhedron (IPP) algorithms.
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Multiplicity of Roots

- **Univariate Case**
  - A root $a$ of $f(x)=0$ has multiplicity $k$ if
    $$f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0, \quad \text{and} \quad f^{(k)}(a) \neq 0$$

- **Bivariate Case**
  - Define
    $$V_f = \{(x, y) \in \mathbb{C} \mid f(x, y) = 0\}$$
    $$V_g = \{(x, y) \in \mathbb{C} \mid g(x, y) = 0\}$$
    - Suppose that $z_0$ is the only common point of $V_f$ and $V_g$ lying above $x_0$. Consider $h(x)=\text{Res}_y(f,g)$, the resultant of $f,g$ with respect to $y$. Then the multiplicity of $z_0=(x_0,y_0)$ as a root of the system is the multiplicity of $x_0$ as a zero of $h(x)$. 
Degree of the Gauss Map

• Let \( p(x,y), q(x,y) \) be polynomials with rational coefficients without common factors, of degrees \( n_1 \) and \( n_2 \), and let \( F=(p, q) \).

• Let \( A \) be a rectangle in the plane defined by \( a_1 \leq x \leq a_2, \ a_3 \leq y \leq a_4 \), \( a_1 < a_2, \ a_3 < a_4, \ a_i \in \mathbb{Q}, \ i=1,2,3,4 \) so that no zero of \( F \) lies its boundary \( \partial A \), and \( p \cdot q \) does not vanish at its vertices.
  - Gauss map \( G: \partial A \rightarrow S^1, \ G = F / \|F\| \), where \( S^1 \) is the unit circle.
  - \( G \) is continuous \( ( \|F\| \neq 0 \ \text{on} \ \partial A ) \).
  - \( \partial A \) and \( S^1 \) carry the counterclockwise orientation.

• Degree \( d \) of \( G \): an integer indicating how many times \( \partial A \) is wrapped around \( S^1 \) by \( G \).
Illustration of the Gauss Map
The Cauchy Index

• Preliminaries
  – $R(x)$: a rational function $q(x)/p(x)$, where $p$, $q$ are polynomials.
  – $[a,b]$ : a closed interval, $a < b$. $R$ does not become infinite at the end points.

• Definition of the Cauchy index
  By the Cauchy index, $I_a^b R$ of $R$ over $[a,b]$, we mean $I_a^b R = N_-^+ - N_+^-$
  where $N_-^+$ ($N_+^-$) denotes the number of points in $(a,b)$ at which $R(x)$ jumps from $-\infty$ to $+\infty$ $(+\infty$ to $-\infty)$, respectively, as $x$ is moving from $a$ to $b$. Notice that $I_a^b R = -I_b^a R$ from the definition.
**The Cauchy Index (continued)**

- **Preliminaries**
  - $A$ : a rectangle defined by $[a_1, a_2] \times [a_3, a_4]$ which encloses a zero.
  - $F = (p, q)$ does not vanish on the boundary of $A$, $\partial A$.
  - $p \cdot q$ is not zero at each vertex of $A$.
  - Let
    \[
    R_1 = \frac{q(a_1, y)}{p(a_1, y)}, \quad R_2 = \frac{q(a_2, y)}{p(a_2, y)}, \quad R_3 = \frac{q(x, a_3)}{p(x, a_3)}, \quad R_4 = \frac{q(x, a_4)}{p(x, a_4)}.
    \]

    Then, we set (for counterclockwise traversal of $\partial A$)
    \[
    I_A F = I_{a_4}^a R_1 + I_{a_3}^a R_2 + I_{a_2}^a R_3 + I_{a_1}^a R_4.
    \]

- **Proposition**

  $I_A F$ is an even integer and the multiplicity
  \[
  d = -\frac{1}{2} I_A F.
  \]
Illustrative Example for Multiplicity Computation Using the Cauchy Index

- \( p(x) = (x-1/2)^5 = 0 \)
- A root of \( p(x) \), \([a] = [0.49,0.51] \).
- \( P(z); (z = x+iy) \)
  
  \[
  p(z) = (x + iy - \frac{1}{2})^5 = f(x, y) + ig(x, y)
  \]
- Create
  
  \( A = [0.49,0.51] \times [-0.01,0.01], \ a_1 = 0.49, a_2 = 0.51, a_3 = -0.01, a_4 = 0.01 \)
- Calculate the Cauchy index
  - Roots of \( f(x, a_3) = 0 \)
  - Calculation of
    
    \[
    I_{a_1}^{a_2} R_3 = -3
    \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Roots of ( f(x, a_3) = 0 ) in [0.1] (from the IPP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.46922316412099, 0.46922316512099]</td>
</tr>
<tr>
<td>2</td>
<td>[0.49273457408967, 0.492734576204823]</td>
</tr>
<tr>
<td>3</td>
<td>[0.49999997363532, 0.500000001889623]</td>
</tr>
<tr>
<td>4</td>
<td>[0.507265424645288, 0.507265426808589]</td>
</tr>
<tr>
<td>5</td>
<td>[0.530776834861365, 0.530776835861365]</td>
</tr>
</tbody>
</table>

- Roots No. 2, 3, and 4 are selected since they lie within the interval \([a]\).
Illustrative Example (Continued)

- Similarly, \( I_{a_3}^a R_2 = -2, \ I_{a_2}^a R_4 = 3, \ I_{a_4}^a R_1 = 2 \)

- Calculate \( I_A F = I_{a_4}^a R_1 + I_{a_3}^a R_2 + I_{a_2}^a R_3 + I_{a_1}^a R_4 = -10 \)

- The multiplicity \( m \) of the root is \( d = -\frac{1}{2} I_A F = 5 \)

Note
- \( I_b^a R = -I_b^a R. \)
- Counterclockwise orientation of \( \partial A \) is assumed.
Direct Computation Method

- Use the map $F$ directly.

\[
\phi_{\text{total}} = \sum_{i=0}^{n} \Delta \phi_{i+1}
\]

\[
d = \frac{\phi_{\text{total}}}{2\pi}
\]
Direct Computation Method

\[ F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x, y) = (f(x, y), g(x, y)). \quad G : \partial A \to S^1, \quad G = \frac{F}{\|F\|}, \]

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Bisection Algorithm for Solving Univariate Polynomial Equations

- Univariate polynomial in complex variable $z$. 
  (Substitute $x$ with a complex variable $z = x + iy$)

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

- Input:
  - initial domain: $S = [a_1, b_1] \times [a_2, b_2]$
  - a complex polynomial: $p(z)$
  - tolerance, number of sample points

- Output
  - real and complex roots, multiplicities

- Algorithm
  - Quadtree decomposition
  - Direct degree computation method: complex interval arithmetic.

Nonlinear Polynomial Systems: Multiple Roots and their Multiplicities
Examples

- Wilkinson polynomial

\[ p(t) = \prod_{i=1}^{20} \left( t - \frac{i}{20} \right) \]

- Complicated Polynomial (degree 22)

\[ p(t) = (t^2 + t + 1)^2(t - 1)^4 \]
\[ (t^3 + t^2 + t + 1)^3(t - 2)(t - 4)^4 \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Multiplicity</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>[-5.956e-10, 5.956e-10] + i[1,1]</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>[1,1] + i[-5.956e-10, 5.956e-10]</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>[4,4] + i[-5.939e-10, 5.939e-10]</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>[-0.5,-0.5] + i[0.866, 0.866]</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>[-1,-1] + i[-5.956e-10, 5.956e-10]</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[-0.5,-0.5] + i[-0.866, 0.866]</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>[-5.956e-10, 5.956e-10] + i[-1,-1]</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>[2,2] + i[-5.94e-10, 0]</td>
</tr>
</tbody>
</table>
Solving a Bivariate Polynomial System

- Change of Coordinates
  - CR: $f$ and $g$ are regular in $y$.
  - CU: whenever two points $(x_0, y_0)$ and $(x_1, y_1)$ satisfy $f = g = 0$, then $y_0 = y_1$.

- Solving a Bivariate Polynomial System
  - Let $f, g$ satisfy CR and CU and let $h(x) = \text{Res}_y(f, g)$. Then the roots of the system $f = g = 0$ are in a one to one correspondence with the roots of $h(x)$. Moreover, $z_i = (x_i, y_i)$ is a real root if and only if $x_i$ is a real root of $h(x)$.
  
  - Let $h(x) = \text{Res}_y(f, g)$ and $l(y) = \text{Res}_x(f, g)$ and $a_{ij} = [t_i, t_{i+1}] \times [s_j, s_{j+1}]$ where in each subinterval $[t_i, t_{i+1}]$ or $[s_j, s_{j+1}]$ there exist precisely one root of $h(x)$ and $l(y)$, respectively. If $a_{ij}$ encloses a real root of $f = g = 0$, then the following must be true
    \[
    0 \in f([t_i, t_{i+1}], [s_j, s_{j+1}]) \times g([t_i, t_{i+1}], [s_j, s_{j+1}])
    \]
Solving a Bivariate Polynomial System: Example

\[
f(x, y) = x^3 - 3x^2 + 5x - 4 + y^3 - 3y^2 + 5y - 2xy = 0,
\]
\[
g(x, y) = 2x^3 - 2x^2 + x - 4 - 4x^2y + 2xy + 9y + 3xy^2 - 8y^2 + y^3 = 0,
\]
\[
h(x) = 56x^9 - 704x^8 + 3880x^7 - 12304x^6 + 24744x^5 - 32736x^4 + 28504x^3 - 15760x^2 + 5024x - 704.
\]
\[
l(y) = -56y^9 + 608y^8 - 2824y^7 + 7312y^6 - 11496y^5 + 11136y^4 - 6328y^3 + 1744y^2 - 32y - 64.
\]

<table>
<thead>
<tr>
<th>Root (x,y)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.999999978, 1.000000001]x[0.99999994, 1.00000001]</td>
<td>5</td>
</tr>
<tr>
<td>[1.57142855, 1.57142859]x[-0.142857209, -0.142857134]</td>
<td>1</td>
</tr>
<tr>
<td>[1.99999999, 2.00000003]x[1.99999996, 2.00000003]</td>
<td>3</td>
</tr>
</tbody>
</table>
Conclusions

• Study of the topological degree and multiple roots of univariate and bivariate polynomial systems in the context of geometric modeling.

• Development of practical algorithms for isolating and computing multiple roots of univariate and bivariate polynomial systems.

• Basis for further research needed in addressing the general problem of single and multiple roots of nonlinear polynomial systems in $n$ variables.